SEMESTER II MATHEMATICS GENERAL

LECTURE NOTE

**INTEGRAL CALCULUS AND DIFFERENTIAL EQUATIONS**

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**SECTION I (Integral Calculus)**

**Text: An Introduction to Analysis: Integral Calculs— K. C. Maity,**

 **R. K. Ghosh**

**CHAPTER 1: DEFINITE INTEGRALS—A REVISION**

Let f be a real valued **continuous** function defined on a **closed** and **bounded** interval [a,b]. Let us choose a **partition** (collection of finite number of points of [a,b] including a and b) P = {a = x0,x1,x2,…,xn=b} of [a,b] (for example: {0,1/2,1} is a partition for [0,1].**How many partitions are there for a given interval—finite or infinite?**).



Fig 1: Area under a curve is approximated by sum of areas of rectangles

Let r = xr-xr-1, r = 1,…,n and r| r = 1,2,…,n}. Choose an arbitrary point cr(xr-1,xr) for all r and consider sum of areas of rectangles . It can be seen that this sum approaches more closely the actual area under the curve if we make width of the rectangles smaller , that is, if we increase number n of points of subdivision (**sum of areas of two rectangles on gives a better approximation to the area under the curve than area of a single rectangle**).

**Definition 1.1 ,** provided the limit exists **independent of choice of points of subdivision xi and that of ci , for all i**. It can be proved that for a **continuous function f defined over a closed bounded interval [a,b], exists in above sense**.

**Simpler equivalent expression for calculating :**

We can make choices of xi and ci **suitably** so as to obtain equivalent simpler expression of .

* Let us choose xi’s equi-spaced, that is , 1 = 2 = … = n = (b-a)/n. Then.
* Let us choose cr = a+rh, r = 1,…,n, where h = (b-a)/n. Then = .

As a special case,

**Example 1.1** From definition, calculate .

**»** = = = 1/3, since nh = 1 holds, for every positive integer n and the corresponding h.

**Fundamental Theorem of Integral Calculus**

**Theorem 1.1** If exists and **if there exists** a function g:[a, b]R such that g1(x) = f(x)(suffix denotes order of differentiation) on [a, b], then = g(b) – g(a).

**NOTE**: g is called a **primitive** of f. A function f may not possess a primitive on [a,b] but may exist ; in that case,can not be calculated using fundamental theorem. Primitives of f on [a, b] are given by the indefinite integral : that is the reason why we consider indefinite integrals.

**Example**  exists, since x2 is continuous on [0,1]. Also g(x)== +c is a primitive of x2 on [0,1]. Hence Fundamental Theorem gives =+c)-c=.

Note is independent of c though involves c.

**PROPERTIES OF DEFINITE INTEGRALS**

We assume below that the definite integrals exist and whenever we consider , a primitive g to f over [a,b] exists, so that we can apply Fundamental Theorem. For a<b, we **define** = - .

1. =+(irrespective of relative algebraic magnitude of a,b,c)

Example +=

+ **=**.



Example =

1. . In particular, if f(a-x) = f(x)for all x in [0,a], then and if f(a-x) = - f(x) for all x in [0,a], then = 0.
2. , if f(a+x) = f(x), n natural.
3. . If f is even,. If f is odd, = 0.

Example =0, = 2

**PRACTICE SUMS**

1. Evaluate:.
2. Evaluate:
3. If f(x) = f(x+ kp) for all integer values of k, show that .

**CHAPTER 2: REDUCTION FORMULA**

In this chapter, we study how to decrease complexity of some integrals in a stepwise manner by the use of recurrence relation that we derive generally using integration by parts formula.

1. Let In = , n natural.

In = = sinn-1x(-cos x)- (n-1)= sinn-1x(-cos x)+(n-1)= -sinn-1x cos x+(n-1)In-2-(n-1)In. hence In=In-2.

If we denote Jn = then Jn = +Jn-2 = Jn-2. By repeated application of the reduction formula, it can be proved that Jn= Jn-2=…==, if n is even natural and Jn ==, if n is odd natural.

1. Let In = , n natural.

Then In = = = .

Also, Jn = =-Jn-2= Jn-2.

1. Let In = , n natural. Then In = = secn-2x tanx – (n-2) = secn-2x tan x – (n-2)(In-In-2). Hence In =+.
2. Let Im,n = = = = = Im,n-2- . Transposing and simplifying, we get a reduction formula for Im,n.
3. Let Im,n = = - = -+ [ since cos nx sin x = sin nx cos x – sin(n-1)x] = -Im,n+. Transposing and simplifying, we get a reduction formula for Im,n­.

**Illustrative examples**

1. = and n>1, show that In+ n(n-1)In-2 = n

 = = n = n-n(n-1). Hence.

1. Im,n = = = (. Hence Im,n can be obtained.
2. Im = . From 5 above, Im = = = . Repeating the use of the reduction formula, it can be proved that Im = .
3. Get a reduction formula for each of the following:,.

**CHAPTER 3: IMPROPER INTEGRAL**

When we consider the definite integral in earlier standards, we implicitly assume two conditions to hold: **(a)** f is continuous on [a, b] or , to that matter, at least the limit exists independent of choice of points of subdivision xi and that of ci, for all i and **(b)** the interval [a,b] is bounded. We want to extend the definition of when either (a) or (b) or both are **not** met. This extended definition of definite integral is referred to as **Improper Integrals**. Improper integrals can be of two types: (a) **Type 1**: interval of integration is unbounded, (b) **Type 2**: integrand has a finite number of infinite discontinuities in the interval of integration.

**Definition of TYPE I improper integral, and**

Let the function f be integrable in [a ,B], for every B>a. If exist finitely, we define = and we say exists or converges; otherwise diverges. Similarly, (provided the limit exists) and , a is any real, provided and exist separately.

**Example 3.1** , . The range of integration of the integrals are unbounded. For a>1, = 1- and = 2(-1). Since = 1 exists but does not exist, hence the improper integral converges and diverges.(compare areas below the curves y = 1/x2 and y = 1/ in diagram below)



Fig.2:Comparison of areas under y=1/x2 and y=1/ in (0,1] and [1,)

**Definition of TYPE II improper integral**

Let f have an infinite discontinuity only at the point a (that is, or and is continuous in (a, b].Then we define ,0<c<b-a, provided the limit exists. Similarly, if f has an infinite discontinuity only at the point b and is continuous in [a, b), then we define , 0<c<b-a, provided the limit exists. If f has an infinite discontinuity at d, a<d<b, and is otherwise continuous in [a,b], we define , provided both of and exist separately.

**Example 3.2** , . The integrands have an infinite discontinuity at x=0. For 0<a<1, and . Since = 2 exists but does not exist, so converges whereas diverges. (Compare areas between x = a,0<a<1, and x = 1, below the curves y = 1/x2 and y = 1/ in diagram above)

**Example 3.3**

The integrand is continuous everywhere but the interval of integration is unbounded. Let a>o be fixed. = = . Thus = = .

**Example 3.4** =.

The integrand has an infinite discontinuity at x = 3 and is continuous on [0, 3). Let 0<a<3. Then = sin-1(a/3) . So = . Hence =.

**Note :** we can apply standard methods of integration, in particular method of substitution, only to a proper integral and not directly to an improper integral. Thus if we substitute z=1/x directly in the improper integral , we get a value -2 of the integral whereas it can be checked from definition that the improper integral diverges.

**PRACTICE SUM**

 = , =(a,b>0),, (5)ln 2.

**TESTS FOR CONVERGENCE OF IMPROPER INTEGRALS**

TYPE I INTEGRAL

**Theorem 3.1** (Comparison test) Let f and g be integrable in [a, B], for every B>a. Let g(x)>0, for all x ≥a. If = c≠0, then the integrals and either both converge or both diverge. If c = 0 and converges , then converges.

**Theorem 3.2**  ( Let f be integrable in [a, B], for every B>a. Then converges if exists with >1 and diverges if exists and ≠0 with ≤ 1.

**Example 3.5**  converges **by comparison test** , since 0≤ for all x0, and converges (need to prove!).

**Example 3.6**  converges by since = 1, = 2>1. Note that is continuous and hence integrable in [1, B]for B>1.

**Example 3.7**  converges **by**  since = 0 (verify using L’Hospital’s rule) , = 2>1 and is continuous, and hence integrable, in [0,B] for B>0.

**Example 3.8**  diverges, since = 1/3, = ½<1 and is continuous, and hence integrable, in [0,B] for B>0.

TYPE II INTEGRAL

**Theorem 3.3** (Comparison test) Let f and g be integrable in [c, b], for every c, a<c<b. Let g(x)>0 , for all x, a<x≤b. If ≠0, then and both converge or both diverge. If = 0 and converges, then converges.

**Theorem 3.4**  ( Let f be integrable in[c, b], for every c, a<c<b. Then converges if f(x) exists for and diverges if f(x) exists (≠0) for ≥1.

**Example 3.9**  converges , since = 1, for <1 and is continuous, and hence integrable, in [c,1]for 0<c<1.

**Example 3.10**  converges, since = 1, for and is continuous, and hence integrable, in [1/2, c]for 1/2<c<1.

**THE GAMMA AND BETA FUNCTIONS**

**Definition** (Gamma function) For n>0, (n) = .

**NOTE**: Gamma function is an improper integral of type I. If 0<n<1, (n) is also an improper integral of type II. We shall assume convergence of the gamma function in our course of study.

**Definition** (Beta function) For m, n>0,(m,n) =

**NOTE**: Beta function is an improper integral of type II if either m or n or both lies between 0 and 1 strictly; otherwise it is a proper integral.

**Properties of Gamma and Beta functions**

1. For any a>0, = (n)/an.

**»** let 0<c<d. consider the proper integral I = . Let y = ax. Then I = = . Thus lim I = as and d.

1. (n+1) = n(n)

 **»** Let 0<c<d. using integration by parts on the proper integral I = , we get I = = (+, which tends to n(n) as c 0+ and d (by use of L’Hospital’s rule). Hence the result.

1. (1) = 1 (can be verified easily)

(n+1) = n, for a natural n (follows from property 2 and 3)

1. (m,n) = (n,m) (follows using a substitution y = 1-x after passing to a proper integral)
2. (m,n) =2(follows using a substitution x = sin2 after passing to a proper integral)
3. (,)= (follows from definition)
4.
5. For 0<m<1, (m) (1-m) = cosec(m

**Example 3.11** .

**»** The range of integration of the given integral is unbounded but the integrand is continuous everywhere. For 0<a, = (substituting y = x2 in the proper integral ). Thus = = = = .

**Example 3.12** .

**»** The integrand has an infinite discontinuity at x = 1.Let 0<c<1. Substituting x3 = sin in the proper integral , = . Since = (5/3,1/3) = = . = .

**PRACTICE SUMS**

1. = .

**CHAPTER 4: DOUBLE INTEGRAL**



Fig. 3:Volume enclosed by the surface z=f(x,y) is approximated by volume of parallelopipeds

Integrals over rectangles

Let f(x,y) be a bounded function of two independent variables x and y defined over a closed rectangular region R: a≤x≤b; c≤y≤d. we take partitions {a = x0,x1,…,xr-1,xr,…,xn = b} of [a,b] and {c = y0,y1,…,ys-1,ys,…,ym = d}. These partitions divides the rectangle R into mn number of subrectangles Rij(1≤i≤n, 1≤j≤m). Let us choose arbitrarily ( and , 1≤i≤n, 1≤j≤m. the volume of the parallelepiped with base Rij and altitude f( is f((xi-xi-1)(yj-yj-1). , sum of the volumes of all the parallelepipeds erected over all of the Rij’s, gives an approximation of the volume enclosed by the curve and the planes x = a,x = b, y = c, y = d and z = 0. The approximation can be improved by increasing number of subrectangles into which R is divided into. Thus the limit , provided it exists, gives the volume and is represented by .

**NOTE:** Every continuous function is integrable over any rectangle.

**Theorem4.1 (equivalence of double integrals with repeated integrals)** If exists over a rectangle R: a≤x≤b; c≤y≤d and exists for each value of y in [c,d], then the repeated integral exists and is equal to

**Example 4.1** Evaluate over R:0≤x≤, 0≤y≤

**#** sin(x+y) is continuous on R, so the double integralexists . Evaluating given double integral in terms of repeated integrals,

 = = = 2.

**Example 4.2** Evaluate over R bounded by y = x2, x = 2, y = 1.



Fig.4:Fixing up of range of integration of independent variables x and y

 = = = = .

**Example 4.3** Prove that = ,where R is the region bounded by y = x and y = x2.

**CHAPTER 5: EVALUATION OF AREA**

**Cartesian co-ordinate**

It has already been seen that area of the region bounded by the curve y = f(x), lines x = a, x = b and y = 0 is given by , provided it exists. Similarly area of the region bounded by the curve x = g(y), lines y = c, y = d and x = 0 is given by , provided it exists. We can define F:[a,b] R by F(t)= ,a≤t≤b.



Observe that for the function f[-2,2]..\Documents\30.gif.



Fig.5: Sign convention of area calculation and area bounded by two curves

**Example 5.1** find thearea of the bounded region bounded bythe curve y = x(x-1)(x-2) and the x-axis.[..\Documents\31.gif](../Documents/31.gif)

**#** y value is positive for 0<x<1 and for x>2 and is negative for 1<x<2; the curve cuts the x-axis at points whose abscissa are 0,1,2. Required area = +.

**Example 5.2** [**..\Documents\x5.mw**](../Documents/x5.mw)

**Example 5.2** Find thearea of the bounded region bounded bythe curves y = x2 and x = y2.

**#** On solving the given equations of the curves, the point of intersection of the two curves are (0,0) and (4,4). Thus required area = .

**Example 5.3** Find the area of the loop formed by the curve y2 = x(x-2)2

**#** The abscissa of points of intersection of the curve with the x-axis are given by y = 0, that is, x = 0,2,2. For x<0, no real value of y satisfy the equation. Hence no part of the curve exist corresponding to x<0. Corresponding to each x-value satisfying 0<x<2, there exist two values of y, equal in magnitude and opposite in sign. Thus between x = 0 and x = 2, the curve is symmetric about the x-axis and a loop is formed thereby. For x>2, y as x. the required area = 2 (by symmetry of the curve about x-axis).

**Example 5.4** Prove that area included in a circle of radius r unit is r2 square unit.

**#** We can choose two perpendicular straight lines passing through the centre of the circle as co-ordinate axes. With reference to such a co-ordinate system, equation of the circle is y = ±.Curve is symmetric about the axes .Thus required area = 4dx.

Polar co-ordinate



Fig.6:Area under a polar curve and that under a cardioide

The area of the region bounded by the curve r = f() , the radius vector , is given by .

**Example 5.5** Find the area enclosed by the cardioide r = a(1+ cos)

**#** As varies from 0 to , r decreases continuously from 2a to a. When further increases from to , r decreases further from a to 0. Also the curve is symmetric about the initial line (since the equation of the curve remains unaffected on replacing by –. Hence the area enclosed by the curve = 2..



Fig.7:Area between two cardioides

**Example 5.6** Find the area enclosed by the cardioide r = a(1+ cos) and r = a(1- cos)

**#** The vectorial angle corresponding the points of intersection of the curves are and =-. Because of the symmetry of the curves about the initial line, ,=- and , required area is 4..

**Notation**: For the rest of our discussion , suffix of dependent variable or of a function denotes order of differentiation.

**CHAPTER 6: LENGTH OF ARC OF A PLANE CURVE**

The length of the arc of a curve y = f(x) between two points whose abscissae are a and b, when f1 (x) is continuous on the interval [a,b] is given by . We can define the arc length function s:[a,b] R by



s(t)=dx ,a≤t≤b. [..\Documents\x6.mw](../Documents/x6.mw) .

 **Example 6.1** Find the length of circumference of a circle of radius 5 units.

**#** With suitable choice of co-ordinates, equation of the curve is x2+y2 = 25. Here 1+y12 = 25/(25-x2). Required length = 4 = 10.

**Example 6.2** Find the length of the arc of the parabola y2 = 16 x measured from the vertex to an extremity of the latus rectum.

**Example 6.3** Let f(x)=x3+1,x[-2,2].In adjoining figure, f is plotted in red, integrand in blue and the arc function s(x)=, x[-2,2], in green.[..\Documents\32.gif](../Documents/32.gif)

**Example 6.3** Find the length of the loop of the curve 3y2 = x(x-1)2.

The length of an arc of the curve r = f between points whose vectorial angles are and is given by .

**Example 6.4** Find the length of the perimeter of the cardioide r = a(1+cos

**#** As we can see from fig.7 above, using symmetry of the curve about the initial line (justified by the invariance of the equation on replacing by –), the required length is 2x.

**CHAPTER 7: SURFACE AREA AND VOLUME OF SURFACE OF REVOLUTION**

The volume generated by revolving about the x-axis an area bounded by the curve y = f(x), the x-axis and two ordinates x = a and x = b is given by . Similarly, The volume generated by revolving about the y-axis an area bounded by the curve x = g(y), the y-axis and two ordinates y = c and y = d is given by . The formula for surface area of the surface generated is given by S= and .

**Example 7.1** Find the volume and surface area of a right circular cylinder of base radius r and altitude h.



Fig.8:Volume and surface area of surface of revolution

**#** A right circular cylinder of radius r and altitude h is obtained by revolving y = r,z = 0 about the x-axis. Thus volume = = r2h and surface area = = rh.

**Example 7.2** Find the volume and surface area of a sphere of radius r.

**#** A sphere of radius r is obtained by revolving y= ,-r≤x≤r, about x-axis. Thus V= and S=2dx=4r2.

**Example 7.3** Surface area of the surface generated by y=cos (x3) between x=0, x=2 for rotation about x-axis and y-axis(frustum view).[..\Documents\33.gif](../Documents/33.gif)

**Example 7.4** Surface area of the surface generated by revolving y=cos x between x=0 and x=2 around x-axis: [..\Documents\x3.mw](../Documents/x3.mw)

**Example 7.5** Surface area of the surface generated by revolving y=1+ cos x between x=0 and x=2 around x-axis:[..\Documents\x4.mw](../Documents/x4.mw)

**SECTION II (ORDINARY DIFFERENTIAL EQUATIONS)**

NOTE: suffix denotes order of differentiation

**TEXT : (1) Differential equations and their applications—Zafar Ahsan**

 **(2) Differential Equations—Richard Bronson (Schaum)**

**CHAPTER 1: ORDER,DEGREE AND FORMATION OF ORDINARYDIFFERENTIAL EQUATION**

**Definition 1.1** An ordinary differential equation(ODE) is an equation involving derivative(s) or differentials w.r.t. a single independent variable.

Example: y2=a2y, 3x3y3+4x2y1 = 7x2+9, x dy – ydx+2xy dy = 0.

**Definition 1.2** the **order** of an ODE is the order of the heighest ordered derivative occurring in the equation. The **degree** of an ODE is the largest power of the heighest ordered derivative occurring in the equation after the ODE has been made free from the radicals and fractions as far as the derivatives are concerned.

**Example**: y = x : first order, second degree

(1+y12)3/2 = y1: first order, sixth degree.

**Definition 1.3** An ODE in which the dependant variables and all its derivatives present occur in first degree only and no products od dependent variables and /or derivatives occur is known as a linear ODE. An ODE which is not linear is called nonlinear ODE. Thus y1 = sin x+cos x is linear while y = y1+1/y1 is non-linear.

**Definition 1.4** A **solution** of an ODEis a relation between the dependent and independent variables, not involving the derivatives such that this relation and the derivatives obtained from it satisfies the given ODE. For example, y = ce2x is a solution of the ODE y1 = 2y, since y1 = 2ce2x and y = ce2x satisfy the given ODE. Some of the solutions of the DE y2=x2-x+1 are depicted as follows: [..\Documents\x1.mw](../Documents/x1.mw) . Some of the solutions y=c cos x+d sin x (c varying between -20 to 10, d between -10 to 20) of y2+y=0 are shown here: [..\Documents\x2.mw](../Documents/x2.mw)

**FORMATION OF ODE**

GEOMETRIC PROBLEMS

**Example 1.1**: Let a curve under Cartesian coordinate system satisfy the condition that the sum of x- and y- intercepts of its tangents is always equal to **a**. Find the ODE corresponding to the curve.

**#** Equation of tangent at any point (x,y) on the curve is Y-y = (X-x). Thus the differential equation expressing the given condition is (x-yx1)+(y – xy1) = a.

PHYSICAL PROBLEMS

**Example 1.2**: Five hundred grammes of sugar in water are being converted into dextrose at a rate which is proportional to the amount unconverted. Form an ODE expressing the rate of conversion after t minutes.

**#** Let y denote the number of grammes converted in t minutes. Then the ODE is = k(500 – y), k constant.

**Example 1.3** A person places Rs.20,000 in a savings account which pays 5% interest per annum, compounded continuously. Let N(t) denotes the balance in the account at any time t. N(t) grows by the accumulated interest payments, which are proportional to the amount of money in the account. The resulting ODE is =0.05 N.

**Example 1.4** Five mice in a stable population of 500 are intentionally infected with a contagious disease to test a theory of epidemic spread that postulates the rate of change in the infected population is proportional to the product of the number of mice who have the disease with the number that are disease free. Let N(t) denote the number of mice with the disease at time t. resulting ODE is = kN(500-N).

**Example 1.5** Newton’s law of cooling states that the time rate of change of the temperature of a body is proportional to the temperature difference between the body and the surrounding medium. Let T denote the temperature of the body and let Tm denote the temperature of the surrounding medium. Resulting ODE is = -k(T-Tm).

ELIMINATION OF ARBITRARY CONSTANTS

Suppose we are given an equation(not a differential equation) containing n parameters (arbitrary constants).by differentiating the given equation successively n times , we get n equations more containing n parameters and derivatives. By eliminating n parameters from the above (n+1) equations and obtaining an equation which involves derivatives upto the n th order, we get an ODE of order n.

**Example 1.6**: Obtain an ODE by eliminating parameters a,b from (x-a)2+(y-b)2 = c2, c constant.

(x-a)+(y-b)y1 = 0, 1+y12+(y-b)y2 = 0. Obtain values of x-a and y-b in the given equation, obtain the ODE.

**Example 1.7**: Find the ODE corresponding to the family of curves y = c(x-c)2 (2.1), where c is a parameter.

Y1 = 2c(x-c) . so y12 = 4c2(x-c)2 (2.2). from (2.1) and (2.2), c = y12/(4y). Substituting in (2.1), required equation is 8y2 = 4xyy1-y12.

**Example 1.8**: Obtain the ODE corresponding to the family of all circles each of which touches the axis of x at the origin.

**#** The equation of an arbitrary circle of the family is of the form (x-r)2+y2 = r2, that is , x2+y2 – 2rx = 0, r is a parameter. Obtain the ODE by eliminating the parameter r.

**GENERAL, PARTICULAR AND SINGULAR SOLUTIONS**

A solution which contains a number of independent parameters **equal** to the order of the ODE is called **the general solution** or **complete integral** of the ODE. A solution obtained from the general solution by putting particular values to at least one of the parameters present in the general solution is called a **particular solution**. A solution which is neither the general solution nor is a particular solution of an ODE is called a **singular solution** of the ODE.

NOTE **(1)** In counting the parameters in the general solution, we must check whether they are independent and are not equivalent to a lesser number of parameters. Thus a solution of the form c1 cosx+c2 sin(x+c3) appears to have three parameters but actually they are equivalent to two; for, c1 cosx+c2 sin(x+c3) = (c1+c2 sin c3) cos x+c2 cos c3 sinx = A cos x +B sin x.

**(2)** The general solution of an ODE may have more than one form, but parameters in one form will be related to parameters in another form. Thus y = c1 cos(x+c2)and y = c3 sin x+c4 sin x are both solutions of y2+y = 0. Here c4 = c1 cos c2,c3 = -c1 sin c2.

**Example 1.9** y = ln x is a solution of xy2+y1 = 0 on (0,∞) but is not a solution on (-∞,) since ln x is not defined on (-∞,0].

**EXERCISES**

1. Find the ODE of the family of circles whose centres are on the y-axis and touch the x-axis.
2. Find the ODE of the family of parabolas whose axes are parallel to the y-axis.
3. A canonical tank of height 20 cm. loses water out of an orifice at its bottom. If the cross-sectional area of the orifice is ¼ cm2, obtain the ODE representing the height h of water at any time t.



**CHAPTER 2: ODE OF FIRST ORDER AND FIRST DEGREE**

An ODE of the first order and first degree is of the form y1 = f(x,y) which is sometimes conveniently written as M(x,y) dx+N(x,y) dy = 0 **(2.1)**, where M(x,y) and N(x,y) are either functions of x,y possessing partial derivatives or constants.

NOTE: First order first degree ODE whose variables are separable or which are homogeneous or which are reducible to homogeneous equations have been studied earlier and will not be repeated here. Students are advised to revise the same.

**Exact equations**

**Definition 2.1** An ODE of the form (2.1) is called exact if and only if there exists a function f(x,y) with continuous partial derivatives such that (2.1) can be written in the form df(x,y) = 0, that is, in the form fx dx+fy dy = 0. In this case, f(x,y) = c will be the general solution of given ODE.

For example, xdx+ydy = 0 is exact since the given equation can be written in the form d(x2+y2) = 0.

**Theorem2.1**  (2.1) is exact iff My = Nx holds.

**Example 2.1** Verify whether the differential equation (sin x cos y+e2x) dx+(cos x sin y+tan y)dy = 0 is exact. Also find the general solution of the equation.

**#** Comparing with the form Mdx+Ndy = 0, M = sin x cos y+e2x, N = cos x sin y+tan y. so My=- sin x siny = Nx. hence given equation is exact. Let f(x,y) = 0 be the solution. Given equation will be identical with fxdx+fydy = 0. Comparing, fx = sin x cosy+e2x(**2.2**), fy = cos x sin y+tan y(**2.3**).From (2.2), f(x,y) = - cos x cos y+1/2 e2x+g(y), where g(y) is the integration constant. Substituting in (2.3), cos x sin y+ g1(y) = fy = cosx sin y+ tan y. Thus g1(y) = tan y so that g(y) = ln sec y. Hence the general solution is f(x,y) = - cos x cos y+1/2 e2x+ln sec y = c, c parameter.

**Integrating factors**

In theory, a non-exact ODE can always be made exact by multiplying the equation by some function (x,y) of x and y. such a function (x,y) is called an **Integrating factor (I.F.)** of the ODE. Although there is always an I.F. for a non-exact ODE, there is no general method of finding the I.F. we shall now discuss methods of finding I.F. in some particular cases .

**Method 1** (by inspection) use is made of exact differentials like

, d(xy) = x dy+y dx ,d, d.

**Example 2.2**: (1+xy)y dx+(1-xy)x dy = 0.

**#** d(xy)+ xy(y dx-x dy) = 0 . thus 0. So = 0. General solution is 1/(xy)+ln(y/x) = c, c parameter.

**Example 2.3**: x dy-y dx = a(x2+y2)dy

d(tan-1(y/x)) = a dy.

**Method 2** if the ODE Mdx+Ndy = 0 is homogeneous and Mx+Ny≠0, then 1/(Mx+Ny) is an IF.

**Example 2.4**: x2y dx – (x3+y3)dy = 0

**#** Here Mx+Ny = -y4≠0 and the given equation is homogeneous. Thus an IF is -1/y4.

**Method 3** if M = y f(xy), N =x g(xy), then 1/(Mx-Ny) is an IF of the ODE Mdx+Ndy = 0.

**Example 2.5**: (xy+2x2y2) y dx+(xy – x2y2) x dy = 0.

**#** Here M = y[(xy)+2(xy)2] and N = x[(xy)- (xy)2]. Thus an IF = 1/[Mx-Ny] = 1/(3x3y3).

**Method 4**  is an IF of Mdx+Ndy =0,if is a function of x.

**Example 2.6**: (x2+y2)dx – 2xy dy = 0

**#** M = x2+y2, N = -2xy, =-2/x. Thus an IF = =e-2 ln x = x-2.

**Example 2.7** (a very useful form of ODE: linear equation in y) y1+P(x) y = f(x), P(x), f(x) are functions of x alone or constants. M = P(x)y-f(x), N = 1. Thus = P(x). An IF = .

**Method 5**  is an IF of Mdx+Ndy =0,if is a function of y.

**Example 2.8** (xy3+y) dx+2(x2y2+x+y4) dy = 0

=1/y. hence an IF = =y.

**Method 6** if the equation Mdx +Ndy = 0 is of the form xayb(my dx+nx dy)+xcyd(py dx+qx dy) = 0, where a,b,c,d,m,n,p are constants,then xhyk is an IF, where h,k are constants and can be obtained by applying the condition that after multiplication by xhyk the given ODE becomes exact.

**Example 2.9** (y2+2x2y)dx+(2x3-xy)dy = 0

**#** Let xhyk be an IF. Multiplying the given equation by this factor, we have (xhyk+2+2xh+2yk+1)dx+(2xh+3-xh+1yk+1)dy = 0 (**2.4**). Since (2.4) is to be exact, from the condition of exactness (My=Nx), we obtain, (k+2)xhyk+1+2(k+1)xh+2yk = -(h+1)xhyk+1+2(h+3)xh+2yk. equating coefficients of xhyk+1and xh+2yk on both sides and solving , we get h = -5/2, k = -1/2. Thus x-5/2y-1/2 is an IF.

**PRACTICE SUMS**

1. (x3ex-my2) dx+m xy dy = 0 (solution: 2x2ex+my2 = c.)
2. (x2y – 2xy2) dx – (x3-3x2y) dy = 0.
3. (x2y2+xy+1) y dx+(x2y2 – x+1)x dy = 0.
4. (y+y3/3+x2/2) dx+[(x+xy2)/4] dy = 0.
5. (xy2-x2)dx+(3x2y2+x2y – 2x3+y2) dy = 0
6. (2y dx+3x dy)+2xy(3y dx+4x dy) = 0.
7. (x3+xy4)dx +2y3dy = 0.
8. (2xy4ey+2xy3+y) dx+(x2y4ey – x2y2 – 3x) dy = 0.

**FIRST ORDER LINEAR DIFFERENTIAL EQUATION**

A differential equation of first order is linear if the dependent variable y and its derivatives occur only in first degree and are not multiplied together. It is of the form y1+Py = Q(**2.5**), where P and Q are either constants or functions of x alone. As we have seen above, is an I.F. of (2.5).

**Example 2.10** (1+x)y1 – xy = 1-x.

Y1 - y = . Comparing with (2.5), P =- and Q=. an I.F. = = . Multiplying given equation by the I.F., we get y(1+x)e-x = = xe-x+c.

**Example 2.11** (1+y2) dx = (tan-1y – x) dy

, which is **linear in x**. an I.F. = exp = . Solution is x=. Put t = tan-1y. The solution is x.

**Example 2.12** (x+2y3) y1 = y

The equation can be put as : x1 – .x = 2y2 (**2.6**), which is **linear in x**. An I.F. = = 1/y. Multiplying (2.6) by the I.F.,x/y = . Thus x = y3+cy.

**PRACTICE SUMS**

1. x cos xy1+y(x sin x+cos x) = 1.
2. Y2+y1 = 0.

**BERNOULLI’S EQUATION: D.E. REDUCIBLE TO LINEAR FORM**

An equation of the form y1+Py = Qyn (**2.7**) where P and Q are constants or functions of x alone and n is constant except 0 and 1, is called a Bernoulli’s equation. Dividing (2.7) by yn and putting v = y-n+1, the equation reduces to v1+(1-n)pv = (1-n)Q , which is **linear in v**.

**Example 2.13** (1-x2)y1+xy = xy2.

Given equation can be written as y-2y1+=. Let y-1 = v, so that v1 = -y-2y1 and the given equation reduces to v1- v = -, which is linear in v. an I.F. = exp= (1-x2)1/2.. Multiplying by I.F. and integrating, v(1-x2)1/2 = (1-x2)1/2+c. thus (1-y) = cy.

Practice sums

1. y1+x sin2y = x3cos2y.
2. xy- y1 = y3
3. (x2y3+2xy) dy = dx
4. Y(2xy+ex) dx – ex dy = 0.

**CHAPTER 3: DIFFERENTIAL EQUATIONS OF FIRST ORDER AND OF DEGREE HIGHER THAN ONE**

An ODE of order one and degree higher than one has the general form

Pn+P1(x,y)pn-1+P2(x,y)pn-2+…+Pn-1p+Pn = 0 **(3.1)** ,where p= dy/dx. In our course of study we will consider solutions of three particular cases of (3.1).

**Case 1** Equations solvable for p

Let us consider the special case when (3.1) is of the form [p-f1(x,y)] [p-f2(x,y)]…[p-fn(x,y)] = 0 **(3.2)**. Equating each factor to zero, we get n number of first order and first degree ODEs. Let their general solutions be f1(x,y,c1) = 0, f2(x,y,c2) = 0,…,fn(x,y,cn) = 0, where c1,c2,…,cn are arbitrary parameters. We observe that f1(x,y,c). f2(x,y,c)…fn(x,y,c) = 0**(3.3)** is a solution of the first order ODE(3.2) and contains one parameter . Thus (3.3) is the general solution of (3.2).

**Example 3.1** x2 p2+xyp-6y2 = 0.

**#** Given ODE is solvable for p: (p-2y/x)(p+3y/x) = 0. General solution of p-2y/x = 0 is y = c1x2 and that of p+3y/x = 0 is xy3 = c2. Hence the general solution of given ODE is (y – cx2)(xy3 – c) = 0, c parameter.

**Case 2** Equations solvable for y

Let us suppose the given ODE (3.1) is expressible in the form y = f(x,p) **(3.4)**. Differentiating w.r.t. x, we get p = g(x,p,dp/dx)**(3.5)**. Solving (3.5), we obtain F(x,p,c) = 0**(3.6)**. Finally, we eliminate p between (3.4) and (3.6). The p-eliminant gives the general solution of (3.4). When the elimination of p cannot be carried out easily, we just leave after remarking that the p-eliminant of (3.4) and (3.6) gives the general solution of (3.4).

**Example 3.2** y+px = x4p2

**#** Differentiating w.r.t. x yields , after simplification, dp/p +2dx/x = 0, which on integration gives p = c/x2 . Substituting this value of p in the given equation , we obtain the general solution xy+c = c2x of the given ODE.

**Case 3** Equations solvable for x

Let us suppose the given ODE (3.1) is expressible in the form x = f(y,p) **(3.7)**. Differentiating w.r.t. y, we get 1/p = g(y,p,dp/dy)**(3.8)**.Solving (3.8), we obtain F(y,p,c) = 0**(3.9)**. Finally, we eliminate p between (3.7) and (3.9). The p-eliminant gives the general solution of (3.7). When the elimination of p cannot be carried out easily, we just leave after remarking that the p-eliminant of (3.7) and (3.9) gives the general solution of (3.7).

**Example 3.3** y2 ln y = xy p+p2

**#** Given equation is x = y ln y/p-p/y, which is in solvable for x form. Differentiating w.r.t. y, after simplification we get dp/p = dy/y, which on integration gives p = cy. Eliminating p , the general solution is ln y = cx+c2.

**PRACTICE SUMS**

1. (p-xy)(p-x2)(p-y2) = 0 [ solution: (lny – x2/2-c)(y-x3/3-c)(x+1/y+c) =0]
2. (p+x+y)(xp+y+x)(p+2x)=0 [ solution(1-x-y-ce-x)(2xy+x2-c)(y+x2-c) = 0]
3. p2+2py cot x = y2[solution: [y(1-cos x)-c][y(1+cos x)-c] = 0]
4. y = yp2+2px[solution:2x+cxy = 0]
5. xp3 = a+bp[solution:p-eliminant of given equation and y = 3a/(2p2)+2b/p+c]

**A USEFUL SPECIAL CASE: CLAIRAUT’S EQUATION**

Clairaut’s equation is of the form y = px+f(p)**(3.10)**. Differentiating w.r.t. x, we get [x+f/(p)] dp/dx = 0. If dp/dx = 0,then p=c (constant). Eliminating p between this and (3.10), the general solution of (3.10) is y = cx+f(c), c parameter. If we eliminate p between x+f/(p) = 0 and (3.10), we get a solution of (3.10) which does not contain any parameter and gives a **singular solution** of (3.10).

**Example 3.4** (y-px)(p-1) = p

**#** y = xp+p/(p-1), in Clairaut’s form. Differentiating w.r.t. x, dp/dx[x-1/(p-1)2] = 0. dp/dx = 0 gives p = c . Thus general solution is y = xc+c/(c-1). From [x-1/(p-1)2] = 0, p = +1. Substituting in given equation, the singular solution is [y-(+1)x].=(+1).

**PRACTICE SUMS**

1. p = ln(px-y) [ general solution:c = ln(cx-y)]
2. y = px+[ singular solution: y2a2(a2-x2) = b2(x2+a2)]

**CHAPTER 4:** APPLICATIONS OF FIRST ORDER ODE

**Example 4.1**  A culture initially has N0 number of bacteria. At t = 1 hour the number of bacteria is measured to be 3N0/2. If the rate of growth be proportional to the number of bacteria present, determine the time necessary for the number of bacteria to triple.

**#** The corresponding ODE is dN/dt = k N subject to N(0) = N0. Separating and integrating, N = N(t) = cekt. At t = 0,N0 = ce0=c. thus N = N0ekt. At t = 1, 3N0/2 = N0ek, so that ek = 3/2. Hence k = ln(3/2) = .4055. So N = N0e.4055t.to find the time when bacteria had tripled, 3N0 = N0e.4055t. Calculating, t = 2.71 hour.[..\Documents\5.gif](../Documents/5.gif)

**Example 4.2** It is foundthat 0.5 per cent of radium disappears in 12 years.(a) What per cent will disappear in 1000 years? (b) What is the half-life of radium?

**#** Let A be the quantity of radium in grammes, present after t years. Then dA/dt represents the rate of disintegration of radium. According to the law of radioactive decay, dA/dt varies as A, that is dA/dt = -kA(since A is decreasing with time). Let A0 be the amount in grammes present initially. Then 0.005A0 g. disappears in 12 years, so that .995 g remains. Thus A = A0 at t = 0, and A = .995A0 g. at t=12(years) . Solution of the ODE is A = ce-kt. Since A = A0 at t = 0, c = A0. So A = A0e-kt. Also at t = 12, A = -995A0, thus -995A0 = A0e-12k, or e-k = (-995)1/12,or k = .000418. Hence A = A0e-.000418t. (a)Now, when t = 1000, A = .658A0, so that 34.2% will disappear in 1000 years. (b) For half-life, A0/2 = A0­ e-.000418t or, t = 1672.1770 years. [..\Documents\6.gif](../Documents/6.gif)

**Example 4.3** A body whose temperature T is initially 200 0C is immersed in a liquid whose temperature is constantly 1000C. If the temperature of the body is 1500C at t = 1 minute, what is the temperature at t = 2 minutes?

**#** According to Newton’s law of cooling, dT/dt = k(T-100). Solving, ln(T-100) = kt+c. when t = 0,T = 200. So c = ln 100. Also , at t = 1,T = 150. So ln 50 = k+ln 100. Hence k = - ln 2. Substituting values of k,c, we get T = 100[1+2-t]. Thus , at t = 2 min, T = 1250C.[..\Documents\7.gif](../Documents/7.gif)

**Example 4.4** A tank contains 100 litres brine in which 10 kg of salt is dissolved. Brine containing 2kg per litre flows into the tank at 5litre/minute. If the well-stirred mixture is drawn off at 4 litre/ minute, find(a) the amount of salt in the tank at time t minute.

**#** Let P(t) denote the number of kg of salt in the tank and G(t), the number of litres of brine at time t. then G(t) = 100+t. P(0) = 10 and G(0) = 100. Since 5(2) = 10 kg of salt is added to the tank per minute and [P/(100+t)]4 kg salt per minute is extracted from the tank, P then satisfies the ODE dP/dt = 10- 4P/(100+t), which is a linear differential equation in P. solving, P(t) = 2(100+t)-190(100)4(100+t)-4.

**Orthogonal Trajectories**

Two families F1 and F2 of curves are mutually orthogonal if each curve of any one family intersects each member of the other family at right angles. We say either family forms a set of **orthogonal trajectories** of the other family.



Fig.9:Orthogonal Trajectories

As an example, consider the family of all circles having their centres at the origin. The orthogonal trajectories for this family of circles would be members of the family of straight lines passing through the origin. Similarly, the family of elipses and the family members of which are drawn in yellow form orthogonal trajectories of each other. As a physical example, the family of lines of force around a bar magnet and the family of equipotential lines form orthogonal family of each other.

 To find the orthogonal trajectories of a given family of curves, we first find the differential equation of the family . In the differential equation we replace dy/dx by -1/dy/dx and solve the resulting differential equation.

**Example 4.5** Find the orthogonal trajectories to x2+y2 = cx.

**#** c = (x2+y2)/x. So y1 = (y2-x2)/(2xy). The family of orthogonal trajectories thus have the differential equation y1 = 2xy/(x2 – y2). Solving, the equation of orthogonal trajectories is x2+y2 = c1y. Thus the given family of curves is self-orthogonal.

**Example 4.6** Find orthogonal trajectories of the family of rectangular hyperbolas xy = c.

**#** The ODE of the family is y1 = -y/x. The ODE of the family of orthogonal trajectories is y1= x/y. Solving y2-x2 = c.



**CHAPTER 5:** LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER

A linear ODE of order n is an equation of the form

an(x)yn+an-1(x)yn-1+…+a1(x)y1+a0(x)y = Q(x)(**5.1**)

where a0,a1,…,an,Q are continuous real functions on an interval I and a0(x)≠0 on I.

**Notation:** ai standing for i th function of x whereas yi standing for ith derivative of y.

If Q(x) is identically zero on I, then (5.1) reduces to an(x)yn+an-1(x)yn-1+…+a1(x)y1+a0(x)y =0 (**5.2**) and is referred to as **homogeneous** ODE of order n. If all the coefficients ai of (5.2) are constants, it is called linear homogeneous equation with constant coefficients. The general solution of (5.2) is called **complementary function**(C.F.) of (5.1).

**Solution of the homogeneous linear differential equations with constant coefficients**

Consider the differential equation (DE) anyn+an-1yn-1+…+a1y1+a0y = 0**(5.3)**, where a0,a1,…,an are all constants and an≠0.

Let y = emx be a solution of (5.3), where m is a suitable constant to be determined suitably. Calculating y1,…,yn, substituting in the equation and noting that emx≠0, we obtain the **characteristic equation**(CE) or **auxiliary equation** anmn+an-1mn-1+…+a1m+a0 = 0. While solving the CE, following three cases may arise:

* all the roots are real and distinct
* all roots are real but some are multiple roots
* some roots are real and some are conjugate complex
* all roots are imaginary.

We next explain through examples how to find general solution of (5.3) under different possible cases.

**Example 5.1** y3+6y2+11y1+6y =0

Roots of auxiliary equation are -1,-2,-3. (Show calculations!). Thus y = e-x, y = e-2x and y = e-3x are solutions of given DE . Solutions are also linearly independent since the **wronskian**

= ≠0 for all x(calculate and **verify**). Hence the general solution is y = c1e-x+c2e-2x+c3e-3x, c1,c2,c3 are independent parameters.

**Example 5.2** y3-3y1+2y = 0.

Roots of auxiliary equation are 1,1,-2. y = ex, y = xex and y = e-2x are linearly independent solutions. General solution is y = (c1+c2x)ex+c3e-2x.

**Example 5.3** y3+y = 0.

Roots of auxiliary equation are -1, 1/2 ±/2 i. General solution is y = c1e-x+ex/2(c2 cosx/2+c3 sin x/2).

**Solution of the nonhomogeneous linear differential equations with constant coefficients by means of polynomial operator**

The general solution of (5.1) is given by y = yc+yp, where yc is the complementary function( that is, the general solution of the corresponding homogeneous equation) and y = yp(x) is a particular integral(**P.I.**, that is, a particular solution) of (5.1). above we have already seen how to find out yc for a constant coefficient homogeneous linear DE. Below we explain how to find PI using **D-operator method**. We use the notation: Dy = y1, D2y = y2,…,P(D) = a0+a1D+a2D2+…+anDn. Using this notation, the linear nonhomogeneous constant coefficients DE of order n: anyn+an-1yn-1+…+a1y1+a0y = Q(x) can be written as P(D)y = Q(x). The operator P(D) has the following properties:

**Property 1** If f(x) is an n th order differentiable function of x, then P(D)(eaxf) = eaxP(D+a)f, where a is a constant.

**Property 2** Let P(D)y = Q(x). The inverse operator of P(D), written as P-1(D) or 1/P(D), is then defined as an operator which, when operating on Q(x), will give the P.I. yp of P(D)y = Q(x) that contains no constant multiples of a term in CF: P-1(D)Q(x) = yp, or, yp=(1/P(D))Q(x).

**Property 3** If P(D)y = eax, then yp = eax = eax/P(a), P(a)≠0.

**Property 4** If P(D)y = sin(ax) or cos(ax), then yp =sin (ax) or cos(ax).

**Property 5** If P(D)y =eaxV(x), then yp = eaxV = V.

**Property 6** If P(D)y =xV(x), then yp = (xV) = x V-.

**Example 5.4** (D2-4)y = x2, y = 0 and = 1 when x = 0.

**#** C.F. yc = ae2x+be-2x

P.I. yp = x2 = x2 = -x2 = - -1 x2 = x2= - .

Thus the general solution is y = yc+yp = ae2x+be-2x- .

Hence y1 = 2ae2x – 2be-2x - . Using the given conditions, a+b = and a – b = . Hence a = 5/16 and b = -3/16. Thus the PI satisfying the given condition is y = 5/16 e2x-3/16 e-2x- .

**Example 5.5** (D2-2D+5)y = e-x

**#** C.F. e-x(c1cos 2x+ c2 sin 2x)

P.I. yp=e-x = e-x = e-x/8.

General solution is y = e-x(c1cos 2x+ c2 sin 2x)+ e-x/8.

**Example 5.6** (D2-3D+2)y = 3 sin 2x

**#** C.F. aex+be2x

P.I. yp = 3 sin 2x = 3sin 2x =- 3sin 2x = -3 sin 2x = sin 2x = .

**Example 5.7** (D2-2D+1)y = exx2.

**#** C.F. (a+bx)ex.

P.I. yp= exx2 = exx2 = ex x2 = exD-2x2 = (exx4)/12.

**Example 5.8** (D2+6D+9)y = 2e-3x.

**#** We can not use property 3 , since P(-3) = 0. Instead we apply property 5.

Yp = (2e-3x) = 2e-3x(1) = 2e-3x(1) = 2e-3x.x2/2 = x2e-3x.

**Example 5.9** (D2+2D+1)y = x cos x

**#** P.I. yp = (x cos x) = xcos x - cos x = cos x = xsin x/2- ½ D2(-sin x+cos x) = x sin x/2 – ½(sin x – cos x).

**Example 5.10** (D2+4)y = cos (2x)

**#** Let y = cos(2x), z = sin(2x). then y+iz = (cos 2x+i sin 2x)= e2ix = e2ix(1)

= e2ix(1) = e2ix(1) = (cos 2x+i sin 2x)x/4i = x sin(2x)/4-i x cos(2x)/4. Equating real part, y = xsin(2x)/4.

**PRACTICE SUMS**

1. y2 – 4y1 = x2.
2. Y2 – y1 – 2y = x2ex
3. Y2 = x2
4. Y3+3y2 – 4y = xe-2x.

CAUCHY-EULER EQUATION

An equation of the form anxnyn+an-1xn-1yn-1+…+a1xy1+a0y = Q(x) **(5.4)** , where ai’s are constants , is called a Cauchy-Euler equation of order n. by undertaking a transformation x = et of independent variable, (5.4) can be transformed to a constant coefficient linear ODE.

**Example 5.11** x2y2+xy1-4y = x2.

**#** Putting x = et, or ln x = t and D1 = d/dt, the given ODE can be written as [D1(D1-1)+D1-4]y = e2t, or (D12-4)y = e2t. C.F. is yc = ae2t+be-2t = ax2+bx-2.

P.I. yp= e2t = e2t(1)= e2t(1) = ¼ e2tt = ¼ x2ln x.

Hence the general solution of given equation is y = yc+yp = ax2+bx-2+1/4 x2 ln x.

**PRACTICE SUMS**

1. x2y2+3xy1+y =
2. x2y2+y = 3x2.
3. x2y2+5xy1+3y = e-x.

