**MATHEMATICS HONOURS SEMESTER II**

**LECTURE MATERIAL**

**INTRODUCTION TO METRIC SPACES**

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**DOWNLOAD SITE: WWW.SXCCAL.EDU**

**Reference: (1) Metric Space—S. Kumaresan**

**(2) Metric Space-M.N.Mukherjee**

Section I: Introduction and Examples of Metric Spaces

In the course of its development, Analysis (Real and Complex) became so complex and varied that even an expert could find his way around in it only with difficulty. Under these circumstances, some mathematicians like Riemann, Weierstrass,Cantor, Lebesgue, Hilbert, Riesz became interested in trying **to uncover the fundamental principles on which all of Analysis rests**. It played a large part in the rise of prominence of Topology, Abstract Algebra and the Theory of Measure and Integration; and when these new ideas began to percolate back through Analysis, the brew which resulted was Abstract Analysis.

As Abstract Analysis developed in the hand of its creators, **many a major theorem was given a simpler proof in a more general setting, in an effort to lay bare its inner meaning**. **Much thought was devoted to analyse the texture of the Real and Complex number systems**, which are the context of Analysis. It was hoped--- and these hopes were well founded—that analysis could be clarified and simplified, and that stripping away superfluous underbrush would give new emphasis to what really mattered from the point of view of underlying theory.

Analysis is primarily concerned with limit processes, so it is not surprising that mathematicians thinking along these lines soon found themselves studying (and generalising) two elementary concepts: that **of a convergent sequence of real or complex numbers and that of a continuous function of a real or complex variable**.

Recall from Semester I Calculus course the interesting properties that a real valued continuous function defined on a closed and bounded interval possesses: the boundedness property, the Intermediate value property, the attainment of optimum values and so on. It can be established through counterexamples that these properties need not hold for a discontinuous function, even if it is defined on a closed and bounded interval. Thus continuity is a crucial property of a real valued function of a real variable.

Recall from Semester I Calculus course the definitions of convergence of a sequence of real or complex numbers and that of continuity of f: RR at cR. A sequence (xn) of real numbers converges to a real number x iff for every >0, there exists natural m such that if n is a natural number m, then . A function f: RR is continuous at c iff for every >0, there exists >0 such that implies . It is noteworthy that both these definitions involve the concept of distance between two real numbers: recall that gives the distance between the two real numbers a and b.

We know that distance between two real numbers a and b is given by . Thus f: RR is continuous at c iff **distance** of f(x) from f(a) can be made less than any preassigned >0, provided **distance** between x and a be made less than a suitable >0, in general dependent on . Thus the fundamental concept of continuity of a real valued function of a real variable is expressed in terms of the more fundamental concept of distance between two real numbers. Note that the definition of continuity and convergence of sequence could also been given without bringing in the binary operation ‘subtraction’’ between two real numbers into picture: if we denote the distance between two real numbers a and b by d(a,b), above definition could also be written as:

**Definition 1/** A function f: RR is continuous at c iff for every >0, there exists >0 such that implies .

**Definition 2/** A sequence (xn) of real numbers converges to a real number x iff for every >0, there exists natural m such that if n is a natural number m, then .

We want to generalize this definition of continuity of f: RR at a to that of function g: XX where X is an arbitrary set and the concept of convergence of sequence in R to an arbitrary set X on which some concept of ‘distance’’ has been predefined . In the light of above discussion, such a generalization would be feasible if we can generalize the concept of distance between two real numbers to the concept of distance between two elements of X. Let us look back proofs of some results of Calculus in an attempt to isolate and crystallize basic properties of distance between two real numbers; once that is done, we will axiomatize distance between two elements of X using those properties.

In many branches of Mathematics—in Geometry as well as in Analysis—it has been found extremely convenient to have available a notion of distance which is applicable to the elements of abstract sets. A Metric Space is nothing more than a non-empty set equipped with a concept of distance which is suitable for the treatment of convergent sequences in the set and continuous functions defined on the set.

**Some basic properties of distance between two real numbers**

Let x,y,z be real numbers. Distance between x and y is given by .

1. d(x,y)=0
2. d(x,y)==0 iff x=y
3. d(x,y)==d(y,x)
4. d(x,y)==d(x,z)+d(z,y)

**[ proof of 4:** If , then . If , then . Thus . Hence **]**

Note the use of above properties in the following proof:

**Theorem** If and , then l = m.

Proof Corresponding to every >0, there exist and >0 such that 0<⟹ and 0<⟹. Let = min{,}>0. Then 0< implies **(1)** 0**(4)** =+<=; which , by arbitrariness of >0, implies =0; hence l-m=0 **(2)**, that is, l=m.

Property 1 tells that distance between two real numbers is non-negative; property 2 that two real numbers are equal iff distance between them is zero. Property 3 tells that distance of x from y is same as distance of y from x. Property 4 tells that length of any one side of a triangle is sum of two other sides of the triangle.

**Note** If distance between two complex numbers z1=(x1,y1) and z2=(x2,y2) be given by d(z1,z2)=, then it can be verified that properties **1-4** given above for real numbers also holds for distance d(z1,z2) between two complex numbers z1, z2. Thus ,as was proved above for sequence of real numbers, it can similarly be proved that if (zn) be a sequence of complex numbers such that lim(zn)=a and lim(zn)=b , then a=b. Actually this result holds good for sequence defined on an arbitrary nonempty set on which a concept of distance satisfying properties **1-4** is defined. Observation like this are at the root of consideration of abstract Metric Space.

We can think about many important properties of distance d(x,y)=between two real numbers x,y which donot have their counterpart in case of distance d(x,y) between two elements x and y of a general set X; for example, if binary operation ‘addition’’ is not defined for elements of X, the property d(x+y,x+z)=d(y,z) for real numbers x,y,z has no counterpart for elements of X. **Important points regarding properties 1 to 4 of distance between two real numbers are that (1) they can be stated for elements of an arbitrary set X on which a concept of distance has been introduced and (2)all the basic properties of the concept of distance required for formulation of concepts of continuity of function and convergence of sequence can be formulated in terms of these properties.**

**Definition 1.1** Let X be a nonempty set. A function d: X X XR is a **metric** or a distance function on X iff d satisfies following properties: for all x,y,zX,

1. **(non-negativity)** d(x,y)0 for all x,yX and d(x,y)=0 iff x=y.
2. **(symmetry)** d(x,y)=d(y,x)
3. **(transitivity)** d(x,z)d(x,y)+d(y,z)

(X,d) is called a Metric Space(MS).

**Note** A Metric Space (X,d) essentially constitutes of a set X and a metric d defined on X; if any one of X or d is altered, new Metric Space may result.

Example 1.1 Consider the set R of real numbers with the usual metric d:R X R ,d(x,y)=, x,y real. We shall refer to it as Ru.

Example 1.2 For a complex number z=(x,y), is the modulus of z. We see that

+w = 2 Re (z) 2=2=. Taking positive square root, . Let us now define distance between two complex numbers u and v by: d(u,v)=. Using above inequality, we see that for complex numbers u,v,w, we have d(u,v)=d(u,w)+d(w,v). Thus triangle inequality holds. Other assumptions for metric may easily be verified.

Example 1.3 Let X=Rn=RXRX…XR (n times Cartesian Product, n natural). For x=(x1,…,xn), y=(y1,…,yn) in Rn, let us define d1(x,y)= , d2(x,y)=, d3(x,y)= max{}

Let us verify the triangle inequality for d1,d2,d3.

**Case 1:(X,d1)** By Minkowski’s inequality, +. Hence

d1(x,y)== d1(x,z)+ d1(z,y).

**Case 2:(X,d2)** d2(x,y)=. Since , for all i=1,…,n (using property of absolute value of real numbers), hence summing over i=1,…,n, we get d2(x,y)== d2(x,z)+ d2(z,y). Note that d2(x,y)=0 ⟹⟹, for all i=1,….,n ⟹xi-yi=0, for all i=1,….,n⟹ (x1,…,xn)=(y1,…,yn) ⟹x=y.

**Case 3:(X,d3)** d3(x,y)= max{}. For 1, max{}+ max{}= d3(x,z)+ d3(z,y). Hence d3(x,y)= max{} d3(x,z)+ d3(z,y).

Example 1.4 Let X be a nonempty set . Define d:X X XR by d(x,y)=0, if x=y; =1, if xy. All the properties other than triangle inequality are obvious; let us prove triangle inequality. Let x,y,zX. If d(x,y)=0, then obviously d(x,y)d(x,z)+d(z,y) hold. If d(x,y)=1, then xy; hence z must be unequal to at least one of x or y; hence at least one of d(x,z) and d(z,y) must be equal to 1 while other is nonnegative; hence d(x,y)d(x,z)+d(z,y) holds.

Example 1.5 Let (X,d) be a MS and define : X X XR by (x,y)=min{1,d(x,y)}. Let us prove (x,y)(x,z)(z,y) for x,y,zX. If any of (x,z) or (z,y) be equal to 1, then nothing remains to prove since (x,y)1 by definition. If none of (x,z) and (z,y) be equal to 1, then (x,z)(z,y)(x,z)(z,y) d(x,y) (since d is a metric on X)(x,y) (from definition of . Other properties of metric can be verified for (X,.

Example 1.6 Let (X,d) be a MS and define : X X XR by (x,y)=. All properties of metric other than triangle inequality are obvious. To prove triangle inequality, we first observe that the function f:[0,R, f(x)= is a mononotonically increasing function on [0,. Since d(x,y) d(x,z)+d(z,y), so f[d(x,y)] d(x,z)+d(z,y)];hence (x,y)===(x,z)+(z,y).

Example 1.7 Let X be the set of all real sequences. We wish to regard two points(that is, real sequences)(xn) and (yn) to be close to each other if their first N terms are equal for some large N. Larger the integer N closer they are: d((xn),(yn))=0, if (xn)=(yn); =, if (xn)(yn). The triangle inequality d(x,z)d(x,y)+d(y,z) certainly holds if any two of x,y,z are equal. So assume xy, yz and xz. Let r= min{i: xi yi}, s= min{i: yi zi},t= min{i: xi zi}. Clearly, tmin{r,s} and hence d(x,z)max{d(x,y),d(y,z)}.

Example 1.8 Let (X,d1) and (Y,d2) be MSs. Define d: (X X Y)X(X X Y)R by d((x1,y1),(x2,y2))= max{d1(x1,x2),d2(y1,y2)). d((x1,y1),(x2,y2))+ d((x2,y2),(x3,y3))= max{d1(x1,x2),d2(y1,y2))+ max{d1(x2,x3),d2(y2,y3)) d1(x1,x2)+ d1(x2,x3) d1(x1,x3); similarly d((x1,y1),(x2,y2))+ d((x2,y2),(x3,y3))= max{d1(x1,x2),d2(y1,y2))+ max{d1(x2,x3),d2(y2,y3)) d2(y1,y2)+ d2(y2,y3) d2(y1,y3); hence d((x1,y1),(x2,y2))+ d((x2,y2),(x3,y3)) max{d1(x1,x3),d2(y1,y3))= d((x1,y1),(x3,y3)). Other axioms can be verified. Thus d is a metric on X X Y, called product metric on X X Y.

Example 1.9 Let (X,d) be a MS and f:XY be a bijection. Define : Y X YR by (y1,y2)=(f(x1),f(x2))=d(x1,x2) [y1=f(x1), y2=f(x2)]. (Y, is a MS.

Example 1.10 If d1,d2 are metrics on a nonempty set X, then d1+d2 and max(d1,d2) are also metrics on X.

Example 1.11 Let (X,d) be a MS and Y ⊂X. Define dY:Y X YR by dY (y1,y2)=d(y1,y2) for all y1,y2Y. (Y, dY) is a MS which is called subspace of the MS (X,d) and dY is called the metric induced by d on Y.

Lemma 1.1 Let f:[0,1]R be continuous with f(t)0 for t[0,1]. Then =0 iff f(t)=0 for all t[0,1].

Proof To prove the nontrivial part, assume =0. If f is not identically zero, since f 0, there exists t0 such that f(t0)>0. Let a=f(t0) and =a/2. Since f is continuous at t0, corresponding to chosen >0, there exists >0 such that f(t)(a/2, 3a/2), for t(t0-,t0+). Using various properties of integrals, we see that , contradiction.

Example 1.12 Let C[0,1] be the set of all real-valued continuous functions defined on [0,1]. For x,y C[0,1], let d1(x,y)= and d2(x,y)=. Let us verify d1 is a metric on C[0,1]. d1(x,y)==0 implies, by Lemma above (since x-y is continuous implies is continuous on [0,1]), that =0,for all t[0,1] and hence x=y on [0,1]. Since for x,y C[0,1], x(t)y(t) for all t[0,1] implies , hence triangle inequality can be verified.

Example 1.13 Let C[a,b] be the set of all real-valued continuous functions defined on [a,b]. For x,y C[a,b], let d(x,y)=sup{}. If d(x,y)=0 for x,y C[a,b], then sup{=0; since 0 for all , so =0 for all , and hence x=y. Let us prove the triangle inequality. sup{}+ sup{} =d(x,z)+d(z,y). Hence d(x,z)+d(z,y) is an upper bound for the set {}. Thus d(x,y)=sup{} d(x,z)+d(z,y). In particular, the set B[a,b] of all real-valued bounded functions defined on [a,b] forms a MS under the metric d.

Lemma 1.2 Let 1<p, q<, =1, c0, d0. Then cd.

Proof We first prove for s,t0, s1/pt1/q; result then follows taking s=cp, t=dq. Result holds if s=t or if one of s or t is zero. Hence we assume 0<s<t. consider the continuously differentiable function f(x)=x1/q, x>0. By Mean Value Theorem, f(t)-f(s)=f/(e)(t-s), for some e(s,t). Thus t1/q-s1/q=(t-s). Since e-1/p<s-1/p, it follows that t1/q-s1/q<(t-s). Hence s1/pt1/q-s<(t-s), that is, s1/pt1/q.

**Holder’s Inequalitity**: Let K stand for R or C. Let X=Kn and for 1, let and for p=, let =max{}. For p>1, let q be such that 1/p+1/q=1.For p=1, take q=. Then for all a,bKn.

Proof Taking x= y= in Lemma 2, we get +. Summing over i=1,…,n, we get . Simplifying, we get 1=1/p+1/q, whence the inequality.

**Minkowski’s inequality** For 1p, , for a,bKn.

Proof case1 p=. =max{} max{}+ max{}.

Case2 1p<. If a=0 or b=0, proof is obvious. Hence we assume neither of a and b is zero. Using Holder’s inequality , we get = ==. Similarly for other term, we have . Hence =. Dividing both sides of the inequality by the positive number and using the fact that p-(p/q)=p(1-1/q)=1 yields the Minkowski’s inequality.

**Holder’s Inequalitity**: Let 1<p,q<, =1. Then , for x,y C[a,b], .

Proof: If or is zero, then x or y is identically zero function on [a,b] and the inequality holds trivially. Let 0 and 0. Let c =, d=. From Lemma 2 above, +. Taking integral on both sides, Holder’s inequality is proved.

**Minkowski’s inequality** For 1<p<, , for x,y C[a,b].

Proof: +=(; cancellation of common non-negative factor leads to the result.

Triangle inequality for obviously follows from Minkowski’s inequality.

Example 1.14 (lp-space, 1) Let lp ={(an): anR or C and } with metric dp defined by dp((xn),(yn))=. Let denote the set of all bounded real or complex sequences with the metric . Let us verify that dp is a metric on lp. Using Minkowski’s inequality, for each natural n, =+. Hence the sequence of partial sums of the series is monotonically increasing and bounded above by ; thus is convergent and the sum of the series is . Hence . Thus . Hence dp((xn),(yn))== dp((xn),(zn))+ dp((zn),(yn)).

Example 1.15 Let X be the set of all real sequences (xn) such that <1 , for all natural n, and let d((xn),(yn))=, for (xn),(yn)X. Prove that (X,d)is a MS.

Example 1.16 Let X be the set of all continuously differentiable functions on [a,b] and d(f,g)= sup{sup{ for f,gX. Verify whether (X,d) is a MS.

Example 1.17 Let X be a nonempty set . Then d: X X XR is a metric on X iff (1) d(x,y)=0 iff x=y, x,yX and (2) d(x,y)d(x,z)+d(y,z), for all x,y,zX hold.

Proof Let d satisfy the conditions (1) and (2).Putting z=x in (2), d(x,y)d(x,x)+d(y,x)=d(y,x). Again, from (2), replacing y,x,y for x,y,z respectively, we get d(y,x) d(y,y)+d(x,y)=d(x,y). Hence d(x,y)=d(y,x). If d(x,y)<0 for some x,y in X, then d(x,x) d(x,y)+d(y,x)=2d(x,y)<0, contradicting (1). Thus d(x,y)0, for x,yX. For x,y,zX, d(x,y)d(x,z)+d(y,z)=d(x,z)+d(z,y). Hence d is a metric on X. Converse part is clear.

Example 1.18 For x,y,a,b in a MS(X,d), prove that .

Show that for all x,yR, d(x,y)= defines a metric on R, and R is bounded in (R,d)( see definition 4.1)

Answer d is a metric follows from the properties of absolute value function. Also , d(x,y)=<+

Example 1.19 Let X be the set of real sequences (xk) such that 0xk for all natural k. Define d(x,y)=. Show that (X,d)is a MS.

Section II: Open Balls and Open Sets

**Definition 2.1** Let (X,d) be a MS. Let xX and r>0. The subsets Bd(x,r)={yX: d(x,y)<r} and Bd[x,r] ={yX: d(x,y) r} are respectively called the **open and closed balls** centred at x with radius r with respect to the metric d. We use this notation only when we want to emphasize that the metric under consideration is d. Otherwise, we denote Bd(x,r) and Bd[x,r] by B(x,r) and B[x,r] respectively. Note that in (X,d), yB(x,r) iff xB(y,r).

Example 2.1 In Ru, B(x,r)={yR: d(x,y)=}={ yR: x-r<y<x+r}=(x-r,x+r). Similarly B[x,r]=[x-r,x+r]. Note that any open interval (a,b), a,bR, is an open ball B. R is not an open ball in Ru: if R= B(x,r)=(x-r,x+r) for some x,rR, then x+r+1R but x+r+1∉(x-r,x+r).

Example 2.2 In (R2,d) where d((x1,y1),(x2,y2))=, the open ball B((a,b),r)={(x,y)R2:<r} = {(x,y)R2:<r2} consists of all points (x,y) inside the circle with centre at (a,b) and radius r.

Example 2.3 In (R2,d1) where d1((x1,y1),(x2,y2))=, the open ball B1((0,0),1)={(x,y)R2:d1((x,y),(0,0))<1} ={(x,y)R2:<1} ={(x,y)R2: x+y<1,-x+y<1,x-y<1,-x-y<1} consists of all points of R2 inside the region bounded by the lines x+y=1,-x+y=1,x-y=1,x+y=-1. Since B1((a,b),r)=(a,b)+r B1((0,0),1), B1((a,b),r) is obtained by shifting the centre of B1((0,0),1) (that is, (0,0)) to (a,b) and dilating by a factor of r (contracting if r<1 and dilating if r>1).

Example 2.4 In (R2,d2) where d2((x1,y1),(x2,y2))=, the open ball B1((0,0),1)={(x,y)R2:d1((x,y),(0,0))<1} = {(x,y)R2: max{}<1}={(x,y)R2:<1, <1} ={(x,y)R2:<1, <1}={(x,y)R2:x<1,x>-1,y<1,y>-1} consists of all points of R2 lying inside the square bounded by x= and y=. Since B2((a,b),r)=(a,b)+r B2((0,0),1), B2((a,b),r) is obtained by shifting the centre of B2((0,0),1) (that is, (0,0)) to (a,b) and dilating by a factor of r (contracting if r<1 and dilating if r>1).

**Note** For anyr>0, there exists s>0 such that any of the balls B((a,b),r), B1((a,b),r) and B2((a,b),r) contains any of B((a,b),s), B1((a,b),s) and B2((a,b),s).

Example 2.5 Let (X,d) be a discrete MS. For r1, xX, B(x,r)={x}, and for r>1, B(x,r)=X. If r<1, B[x,r]={x},for r1, B[x,r]=X. This MS also provides a counterexample to validate that B(x,r)=B(y,s) need not necessarily imply x=y and r=s in a MS: for x,y, xy, B(x,2)=X=B(y,3) though neither xy, 23.

Example 2.6 In the MS (C[0,1],d1) of Example 1.12, let f(t)=0, for all t [0,1]. The ball B1(f,1)={gC[0,1]: consists of all functions g such that area under between o and 1 is equal to 1 unit. Note that g B1(f,1), h∉ B1(f,1) where g(x)= 2x and h(x)=x, x[0,1].

Example 2.7 In the MS (C[a,b],d) of Example 1.12, let f(t)=0, for all t [a,b]. The ball

B (f,1)={gC[0,1]: {sup{<1} = {gC[0,1]: -1<g(t)<1, for all t [a,b]} consists of all continuous function g on [a,b] whose graph in [a,b] lies between y=-1 and y=1.

Example 2.8 In the MS l2, **0**= (0,0,…)l2. The open ball B(**0**,1)={**x**l2:<1}consists of all real sequences (xn) such that <1. Note that (1,0,0,…) ∉ B(**0**,1) whereas (1/2,0,0,…) B(**0**,1). Note that if (xn) B(**0**,1), then B(**0**,1), for p>1.

**Theorem 2.1** (Hausdorff Property of MS) Let (X,d) be a MS and x,y X, xy. Then there exists r>0 such that B(x,r)B(y,r)=.

Proof Since xy, d(x,y)>0. Let r= d(x,y)>0. If zB(x,r)B(y,r), then d(x,y)d(x,z)+d(z,y)<r+r=d(x,y), contradiction. Hence B(x,r)B(y,r)=.

**Theorem 2.2** Let (X,d) be a MS and let A⊂X. Let aA.Then BA(a,r)={xA:dA(a,x)<r}=B(a,r)A, where B(a,r)={xX:d(a,x)<r}.

Proof B(a,r)A={xX:d(a,x)<r}A ={xA:d(a,x)<r}={xA:dA(a,x)<r} [ for a,xA , d(a,x)=dA(x,a)] = BA(a,r).

Example 2.9 In the MS Ru, consider the subspace Z with induced metric. Let xZ and 0<r1. Then BZ(0,)=B(0,)Z = (-Z ={0}. Thus every singleton subset in Z is an open ball in the subspace Z whereas no singleton subset of Ru is an open ball in Ru.

**Definition 2.2** Let (X,d) be a MS and let UX. U is open in (X,d) iff for each xU, there exists r>0 such that B(x,r)U. Note that the null subset of X does not contain any element and hence is vacuously open. On the other hand, for any xX, B(x,r)X for any r>o; hence X is open in (X,d).

Example 2.10 In the MS Ru, any open interval (a,b) is open in Ru : if a=b, nothing remains to be proved; if a<b, and if c(a,b), letting r=min{c-a,b-c}, c(c-r,c+r)(a,b).

Example 2.11 In a discrete MS (X,d), every subset A of X is open in (X,d): if A=, vacuously; if A and aA, then a{a}=B(a,1/2)A.

**Theorem 2.3** Every open ball in a MS (X,d) is open in (X,d).

Proof Let B(x,r) be an open ball in (X,d), xX, r>0. Let yB(x,r). Thus d(x,y)<r. Let s= r-d(x,y)>0. We shall prove that B(y,s)B(x,r). Let z B(y,s). Then d(y,z)<s=r-d(x,y). Hence d(x,z)d(x,y)+d(y,z)<r so that z B(x,r). Hence B(y,s)B(x,r), which, by arbitrariness of yB(x,r), shows that B(x,r) is open subset of (X,d).

**Theorem 2.4**  Let (X,d) be a MS . Then (1) union of an **arbitrary family** of open subsets in (X,d) is open in (X,d), (2) intersection of a **finite family** of open sets in (X,d) is open in (X,d), (3) a nonempty subset G of X is open in (X,d) iff G is expressible as union of open balls.

Proof (1) Let U={} be a family of open subsets in (X,d) and let x. Then x, for some . Since is open , there exists r>0 such that B(x,r). So, x B(x,r). Hence is open in (X,d).

(2) Let U1,…,Un be open subsets of (X,d). If = ,then is open vacuously. Let . Let x. Then x, i=1,…,n. Since each is open (i=1,…,n), so there exists ri>0 such that B(x,ri), i=1,…,n. Let r=min{r1,…,rn}>0. Then B(x,r), for all i=1,…,n. Hence x B(x,r). Thus is open in (X,d).

(3) If G is expressible as union of open balls, then G is open by part (1), since every open ball is an open set in (X,d). Conversely let G be open in (X,d). By definition ofopen subsets in a MS, for every gG, there exists open ball B(g,rg) centred at g such that g B(g,rg)G. Allowing g to vary over G and taking union of all the open balls thus obtained, we have G. Conversely, for h, h; hence G. Combining, G

NOTE Intersection of infinitely many open sets in a MS need not be open in the MS. In Ru, is an open interval, is an open ball and hence is an open set, for every natural n; but ={1} is not open in Ru.

Example 2.12 Let (X,d) be a MS and aX. Fix r>0. The set O={xX: d(x,a)>r} is open in (X,d).

Proof Let yO. Then d(y,a)>r. Let s=d(y,a)-r>0. We shall prove B(y,s)O. Let z B(y,s). Then d(z,y)<s. If d(z,a)r, then d(a,y)<d(z,y)+d(z,a)<s+r= d(y,a)-r+r=d(y,a), contradiction. Hence, by Law of Trichotomy of real number system, d(z,a)>r. Thus zO. Hence B(y,s) )O. Thus O is open.

Example 2.13 Let U=R-Z. Is U open in Ru?

Answer Let aU. Then aZ. a([a],[a]+1) and ([a],[a]+1)Z = . Thus a([a],[a]+1)U. Hence U is open in Ru.

Example 2.14 Let U={(x,y)R2:xy0}. Prove that U is open in R2.

Proof Let (a,b)U. Then a0 and b0. Let r= min{>0. We need show that if (x,y)B((a,b),r), then neither x=0 nor y=0. If y=0, then <r, contradiction. Similarly if (x,y)B((a,b),r), then x0. Hence B((a,b),r)U. Hence U is open in R2.

Example 2.15 Let U={(x,y)R2:x2+y21}. Prove that U is open in R2.

Proof By example 2.2, V={(x,y)R2:x2+y21} is open. W={(x,y)R2:x2+y21}, being an open ball, is open. Thus U=VW is open.

Example 2.16 Let U={(x,y)R2:x>0,y>0}. Prove that U is open in R2.

Example 2.17 Let U={(x,y)R2:x,y}. Prove that U is open in R2.

Proof Let (a,b)U. Then a and b are not integers. Let r= min{a-[a],[a]+1-a,b-[b],[b]+1-b}>0. It can be seen that B((a,b),r) ⊆U. Hence U is open.

Example 2.18 Let A be any **finite** set in a MS (X,d). Prove that X-A is open in (X,d).

Answer Let A={a1,..,an}⊂X. If A=X, then X-A is open. Let AX and let cX-A. Let 0<r< min{d(a1,c),…,d(an,c)}. To prove B(c,r) ⊆X-A. Let yB(c,r). Then d(y,c)<r implies d(y,c)<d(ai,c) for all i=1,…,n and hence yai for i=1,…,n. Thus yX-A. Hence B(c,r) ⊆X-A. Thus X-A is open.

Corollary Every subset of a finite MS (X,d) is open in (X,d).

Example 2.19 Show that a rectangle of the form (a,b)x(c,d) is open in R2 (with usual metric).

Answer Let (m,n)(a,b)x(c,d). Then a<m<b,c<n<d. Let r,s>0 be such that a<m-r<m<m+r<b, c<n-s<n<n+s<d. Let t=min{r,s}. If (x,y)B((m,n),t), then <tr implies x(a,b); similarly y(c,d). Thus B((m,n),t) ⊆ (a,b)x(c,d). Hence (a,b)x(c,d) is open in R2.

Example 2.20 Is Q open in Ru? Is R-Q open in Ru?

Answer Let aQ. For any r>0, the open ball B(a,r)=(a-r,a+r) contains irrational number (**by density of irrational numbers in R**) and hence B(a,r)⊄Q. Hence Q is not open in Ru. Similarly R-Q is not open in Ru.

Example 2.21 Consider the MS C[0,1] under sup norm. Let E={fC[0,1]: f(0)0}. Is E open in C[0,1]?

Answer Let fE. Thus f(0) )0. Let r=>0. We show that B(f,r)={gC[0,1]: d(f,g)=sup{[0,1]}<r}⊂E. Let g B(f,r). Then d(f,g)=sup{[0,1]}<r. Hence , in particular, <r. Thus <r= implying ; hence g(0) 0. Hence B(f,r) ⊆E, proving E is open in C[0,1].

Example 2.22 Show that the open ball B1(0,1)={fC[0,1]:} in the MS(C[0,1], ) where d1(f,g)= , is open in (C[0,1], ) where =sup{[0,1]}.

Let f B1(0,1). We shall prove B1(0,1). Let g. Then sup{[0,1]}<. Then sup{[0,1]}(1-0) <. Hence (?)

Example 2.23 Let X,Y be MSs. Show that any set of the form B(x,r) x B(y,s) X X Y is open in X XY.

Proof Let (a,b) B(x,r) x B(y,s). dX(a,x)<r and dY(b,y)<s. Let t= min{r- dX(a,x),s- dY(b,y)}. We shall prove B((a,b),t) B(x,r) x B(y,s). Let (u,v) B((a,b),t). Then max{dX(u,a),dY(v,b)}=d((u,v),(a,b))<t. Hence dX(u,a)<t r- dX(a,x) implying dX(u,x)<r; similarly dY(v,y)<s. Thus uB(x,r), vB(y,s) implying B((a,b),t) B(x,r) x B(y,s). Hence B(x,r) x B(y,s) X X Y is open in X XY.

Example 2.24 Let X,Y be MSs. If U be open in X and V be open in Y, then U X V is open in X X Y( with product metric)

Proof Let (a,b)U X V. Since U,V are open and aU, bV, there exists r,s>0 such that B(a,r)U, B(b,s). (a,b) B(a,r)X B(b,s) and B(a,r)X B(b,s) is open in X X Y(Example 2.21).Thus there exists t>0 such that (a,b)B((a,b),t) B(a,r)X B(b,s) U X V. Hence U X V is open in X X Y.

Example 2.25 Let X,Y be MSs. Let W be open in the product metric space X X Y. Let pX : X X YX and pY:X X YY be the projection maps defined by pX(x,y)=x and pY(x,y)=y. Prove that pX(W) is open in X and pY(W) is open in Y.

Proof Let a pX(W). Then there exists bY such that (a,b)W. Since W is open in the product MS X X Y, there exists t>0 such that B((a,b),t)W. Thus a pX[B((a,b),t)] pX(W). Next we prove BX(a,t) pX[B((a,b),t)]. u BX(a,t)⟹dX(u,a)<t⟹max{ dX(u,a),dY(b,b)}<t⟹d((u,b),(a,b))<t⟹(u,b)B((a,b),t)upX[B((a,b),t)]. Hence there exists t>0 such that a BX(a,t) pX[B((a,b),t)] pX(W). Thus pX(W) is open in X.

Example 2.26 Let (X,d) be a MS and let be the metric defined on X by (x,y)= min{1,d(x,y)}. Prove that a subset U of X is d-open iff U is -open.

Proof From definition of , (x,y)d(x,y), for x,yX. Let U be -open. Let uU. There exists r>0 such that (u,r)U. Now ={xX: (x,u)<r}**⊂**{xX: (x,u)<r}=(u,r)U. Thus U is d-open. Conversely, U be -open and let 0<r<1. (u,r)={ xX: (x,u)= min{1,d(x,y)}<r<1}={ xX: (x,u)=d(x,u)<r}=(u,r). If r>1, then (u,)Bd(u,)Bd(u,). Combining, for every r>0 and for every uU, there exists s>0 such that (u,)Bd(u,). Hence U is -open.

**Definition 2.3**  Two metrics d 1 and d2 on a set X are equivalent iff given any x , any open ball (x,r) contains a ball (x,s) for some s>0 and any open ball (x,t) contains a ball (x,u) for some u>0 .

**Theorem 2.5** Two metrics d 1 and d2 on a nonempty set X are equivalent iff there exist r,s>0 such that rd1(x,y)d2(x,y)sd1(x,y), for all x,yX.

Proof Let d 1 and d2 be equivalent metrics on X and, if possible, there exists a,bX such that d2(a,b)<rd1(a,b), for every r>0. Then (a, 2d2(a,b)) does not contain (a,u) for any u>0

Example 2.27 Example 2.23 shows that d and are equivalent metrics.

Example 2.28 Show that the metrics d,d1 and d2 on Rn (discussed in Examples 2.2 -2.4) are equivalent. In fact, we have d2(x,y)d(x,y)d2(x,y), d2(x,y)d1(x,y)n d2(x,y) for all x,yRn.

Example 2.29 Show that the metrics d1 and on C[0,1] defined by d1(f,g)= and (f,g)=sup{} are not equivalent.

Example 2.30 Let (X,d) be a MS and Y be a set. Assume that f: XY is a bijection. We can transfer the metric on X to Y in an obvious way: (y1,y2)=d(x1,x2) where y1=f(x1), y2=f(x2). is well-defined on Y. A concrete example is f:[1,)(0,1] , f(x)=. Thus we get a new metric on (0,1] by setting (x,y)=. It can be shown that is equivalent to the standard metric on (0,1].

Example 2.31 Let f: [0,)0,) is continuous function with the following properties: (a) f(t)=0 iff t=0, (b) f is non-decreasing on 0,), (c) f is subadditive: f(x+y)f(x)+f(y) for all positive x and y. If d is a metric on a set X, then f0d is also a metric on X. The metrics d and f0d are equivalent. As a specific example, consider f(t)=.

**Definition 2.4**  Let (X,d) be a MS and SX. xS is an interior point of S if there exists r>0 such that B(x,r)S. The set of interior points of S is denoted by S0.

**Theorem 2.6** Let (X,d) be a MS . Let AX. Then (1) A is open iff A=A0, (2) A0 is the largest open set contained in A (3) A⊆B⟹A0⊆B0, (4) , (5)A0B0, equality may not hold.

Proof (2) Let xA0. There exists rx>0 such that B(x,rx)⊆A. Since B(x,rx) is an open set, for each yB(x,rx), there exists sy>0 such that B(y,sy)⊆ B(x,rx)⊆A. Hence yA0 and consequently B(x,rx) ⊆A0. Since xA0 is arbitrary, it follows that A0 is open.

Let U be an open set and U⊆A. Let xU. Since U is open, there exists r>0 such that B(x,r) ⊆ U⊆A. Thus xA0. Hence U⊆ A0. Thus A0 is the largest open set contained in A.

(3) A ⊆B⟹A0⊆ A ⊆B⟹ A0⊆B⟹ A0⊆B0.

(4) ⊆A,⊆B; by (3),⊆A0,⊆B0. Hence .

Conversely, let x. Then there exists r,s>0 such that B(x,r) ⊆A and B(x,s) ⊆B. Let t= min {r,s}>0. Then B(x,t) ⊆A.Hence x. Thus ⊆. Combining, we get the result.

(5) follows from (3). In Ru, taking A=Q, B=R-Q, we get the required counterexample.

Example 2.32 Find (1)Q0 in Ru, (2)(0,1]0 in Ru, (3) A0 in R2, where A={(x,y)R2: (x-0)2+(y-0)21}.

Example 2.33 Let A={(x,y):x0,y0} be endowed with the induced metric as a subset of R2 with the Euclidean metric. Draw B(A,d)(0,1).

**Theorem 2.7** Let (X,d) be a MS and U⊆Y⊆X. U is open in (Y,dY) iff there exists an open set V in X such that U=YV.

Proof Let U be open in (Y,dY). For yU, there exists ry>0 such that BY(y,ry)⊆U. As y varies over U, we have U==V, where V= is open in X.

Conversely , let U= V, where V is open in X. Then , for any yU, yV and yY. Since V is open in X, there exists r>0 such that B(y,r)⊆V. Thus BY(y,r)=YB(y,r) ⊆YV=U. Thus U is open in (Y,dY).

Example 2.34 Let Y={(x,y)R2: x0,y0} in R2. Let A={(x,y)Y: 0x<1, 0y<1}. Is A open in Y?

Example 2.35 Which subsets of Z are open in the metric subspace Z of the MS Ru?

Example 2.36 Let X be a MS and Y⊆X be open in X. Prove that Z ⊆Y is open in Y iff Z is open in X. Give counterexample to establish that the result may not hold if Y is not open in X.

Example 2.37 Let Y=[0,2](5,6) in Ru. Since [0,2]=(-1,3)Y, [0,2] is an open ball and hence open set in Y; similarly (5,6)=(5,6) Y is open in Y. Since Y=[0,2](5,6) and [0,2](5,6)=, each of [0,2] and (5,6) are also closed in Y.

Example 2.38 In the metric subspace N of Ru, for each nN, {n}=(n-1,n+1)N; hence {n} is open in the subspace N; thus every subset of N is open in N.

Example 2.39 R , considered as a subspace of R2 with usual metric, is identical to Ru.

Section III: Closed Set

**Definition 3.1** A subset C of a MS (X,d) is closed in (X,d) iff X-C is open in (X,d).

Example 3.1 A=[0,1] is closed but B=[3,4) is not closed in Ru. Let xR-A. Then x<0 or x>1. If x<0, then x(x-, x+)⊆R-A. If x>1, then x(1,x+1)⊆R-A. Thus R-A is open in Ru and hence A is closed in Ru. 4R-B, but there does not exist r>0 such that (4-r,4+r)⊆R-B; hence R-B is not open and as a result B is not closed in Ru.

Example 3.2 Since in a MS (X,d), both and X are open, X and are closed in (X,d).

Example 3.3 Any finite subset in a MS (X,d) is closed in (X,d), by Example 2.17.

Example 3.4 Any subset in a discrete MS (X,d) is open in (X,d) and hence every subset of X is closed in (X,d).

Example 3.5 Let A={(x,y)R2:xy=0}. A is closed in (R2,d) (d stands for the usual metric), by Example 2.13.

Example 3.6 Let C={(x,y)R2:x2+y21}. By Example 2.14, C is closed in (R2,d) (d stands for the usual metric in R2).

Example 3.7 Since Q and R-Q are not open in Ru, Q and R-Q are also not closed in Ru.

Example 3.8 Prove that arbitrary intersection and finite union of closed sets are closed in a MS. Give counterexample to establish that arbitrary union of closed sets in a MS need not be closed in the MS .

Example 3.9 Prove that a plane P={(x,y,z)R3: ax+by+cz+d=0, a,b,c,d are reals} is closed in R3(usual metric).

Example 3.10 Prove that a closed ball B[a,r] in a MS (X,d) is a closed set in (X,d).

Proof Let bX- B[a,r]={xX: d(x,a)>r}. Thus d(a,b)>r. Let s= d(a,b)-r>0. We shall show B(b,s)⊆ X- B[a,r]. If c B(b,s), then d(a,b)d(a,c)+d(c,b)s=d(a,b), contradiction. Hence B(b,s)⊆ X- B[a,r] and thus B[a,r] is closed in (X,d).

Example 3.11 Prove that the set D= {xX: d(x,a)=r} is closed in (X,d).

Proof C={xX: d(x,a) r}=X-B(a,r) is closed since B(a,r) is open in (X,d). Thus D=C B[a,r] is closed in (X,d).

Example 3.12 Show that there exists closed sets F and C in R2 (usual metric) such that F+C={f+c:f is not closed in R2.

Answer Let F={(x,y)R2:xy1} and C={(x,y)R2: x=0}.

Example 3.13 Let A be a nonempty subset of a MS (X,d). Prove that BA is closed in the subspace A iff there exists closed set F of X such that B=FA.

Example 3.14 Show that {xQ:-1<x<1} is open in Q but not closed in Q and that { xQ:-} is both open and closed in Q.

**Definition 3.2** Let (X,d) be a MS and let E⊆X, bX. b is a **limit point** of E iff B(b,r)[E-{b}], for every r>0. Thus, b is a limit point of E iff every open ball centred at b intersects E in at least one point other than b. If bE and b is not a limit point of E , then b is an **isolated point** of E: if bE is an isolated point then there exists r>0 such that B(b,r)E={b}. The set of all limit points of E is called the **derived set** of E and is denoted by E/. E/ is called **closure** of E in (X,d).

**Theorem 3.1**  Let x be a limit point of E in (X,d). Then B(x,r)E is an infinite set, for r>0.

Proof let r>0 be given. Since x is a limit point of E, there exists x1B(x,r)[E-{x}]. Let d(x,x1)=r1. Then 0<r1<r (since x1x, d(x,x1)>0. Since x1B(x,r), d(x,x1)<r). By definition of limit point, there exists x2B(x,r1)[E-{x}]. Then x2x, x1 and x2B(x,r1)B(x,r). Choose B(x,r2), where r2=d(x,x2). There exists x3 B(x,r2) [E-{x}] with x3x1,x2 and x3B(x,r). Proceeding in this way, we get infinitely many distinct points x1,x2,… in EB(x,r).

Corollary A finite set in a MS cannot have a limit point.

Example 3.15 Q/=R,(R-Q)/=R, Z/=N/=. This gives example of an infinite set having no limit points.

**Theorem 3.2**  A subset F of a MS (X,d) is closed in X iff F/⊆F.

Proof If F is closed, then X-F is open in X. If xX-F, then there exists r>0 such that B(x,r)⊆X-F; thus B(x,r)F=; x is not a limit point of F. Thus F/⊆F.

Converesly , let F/⊆F. let xX-F. Then x is not a limit point of F. Hence there exists r>0 such that B(x,r)F=, that is, B(x,r) ⊆X-F. Thus X-F is open and hence F is closed.

Example 3.16 Let E be a nonempty bounded and closed subset of Ru. Show that sup E, inf E E.

Proof Since E is bounded, sup E and inf E exists as real numbers. If E is finite, sup E =max E and inf E = min E obviously E. If E is infinite, since sup E and inf E are limit points of E and E is closed, sup E, inf E E.

**Theorem 3.3**  Let (X,d) be a MS and C⊂Y⊂X. C is closed in (Y,dY) iff there exists a closed set F in X such that C=YF.

Proof C is closed in (Y,dY)⟺ Y-C is open in (Y,dY) ⟺Y-C=YO, O open in X

⟺C=Y-( YO)=(XY)(YO)=Y[X-O]=Y F, where F=X-O is closed in X.

**Theorem 3.4**  Let (X,d) be a MS and E⊆X. Then is the smallest closed set containing E in (X,d).

Proof We shall prove X- is open in (X,d). Let x X-. Then xE and xE/. Thus there exists r>0 such that B(x,r)E= (since x is not a limit point and xE). Also B(x,r)E/=; for ,otherwise,if z B(x,r)E/, then B(x,r),being an open set, is an open neighbourhood of z and hence B(x,r)E(since zE/). Hence B(x,r)E/]=, that is, B(x,r)⊆X-. Hence X- is open and so is closed.

Clearly E⊆. Let F be a closed set containing E. Let xE/. For each r>0, B(x,r)[E-{x}]. Thus B(x,r)[F-{x}]. Thus xF/⊆F (since F is closed). Thus E/⊆F. Hence is the smallest closed set containing E.

**Theorem 3.5**  Let (X,d) be a MS and A,B⊆X. Then (1) , (2) , equality may not hold in general.

Proof (1) First we prove ()/=A/B/. Since A,B⊆AB, hence A/,B/⊆( AB)/ and thus A/ B/⊆( AB)/. Conversely, x∉ A/ B/ ⟹ ther exists r,s>0 such that B(x,r)[A-{x}]=, B(x,s)[B-{x}]=. Let t= min{r,s}. Then B(x,t)[A-{x}]=, B(x,t)[B-{x}]=, hence B(x,t) =. Hence x∉()/. Thus ()/=A/B/. Hence

=((/ =(( A/B/)=(A A/)B/)=.

(2) Since A and B, hence . Counterexample: Take A=Q, B=R-Q in Ru.

Example 3.17 Show that in a MS ⊆B[x,r]. Give an example to show that can be a proper subset of B[x,r].

Answer Since B[x,r] is closed and B(x,r) ⊆B[x,r], ⊆B[x,r]. In a discrete MS (X,d), =={x}X=B[x,1], supposing X is not singleton {x}.

Example 3.18 Show that the diagonal ={(x,x):xX} is closed in the product MS X X X.

Proof Let (u,v). Then uv. There exists r>0 such that B(u, r)B(v, r)=. We shall prove B((u,v),r)=. Let (a,b) B((u,v),r). Then d(a,u) max{d(a,u),d(b,v)}=d((a,b),(u,v))<r; similarly d(b,v)<r; hence a=b B(u, r)B(v, r)=, contradiction.

Example 3.19 Show that the only nonempty subset of R which is both open and closed in Ru is R.

[Hint: If A is a nonempty subset of R which is both open and closed , let x. Then (x-r,x+r) ⊆A, for some r>0. Let m=sup{b:(x-r,b) ⊆ A, b>x}. Why cannot m be finite? If m were finite, conclude that mA].

Example 3.20 If U,V are subsets of a MS (X,d) and if U is open in (X,d),prove that (1) UV=, (2).

Example 3.21 Let a be a point in a MS (X,d) and 0<r<s. Prove that the set {xX: r<d(x,a)<s} is open in (X,d).

Example 3.22 Show that the sets A=N, B={n+} are closed and disjoint in Ru. Find d(A,B).

Section IV: Bounded Set

**Definition 4.1** A subset A in a MS (X,d) is bounded in (X,d) iff there exists x0X and r>0 such that A⊆B(x0,r).

Note Every (open and closed) ball in a MS is bounded in the MS.

Example 4.1 Every finite subset of a MS is bounded in the MS.

Example 4.2 An infinite subset of a MS may or may not be bounded in the MS: (0,1) is bounded in Ru while N is unbounded in Ru.

Example 4.3 Every subset in a discrete MS (X,d) is bounded in (X,d): if A⊆X and x0 be any fixed element of X, then A⊆B(x0,2).

Note A subset A of a MS (X,d) is bounded iff for every xX there exists rx>0 such that A⊆ B(x,rx). If for every xX there exists rx >0 such that A⊆ B(x,rx), then obviously A is bounded. Conversely , let A be bounded. Then there exists x0X and r>0 such that A⊆B(x0,r). Let yX. Let ry=r+d(x0,y). For aA, d(a,y)d(a,x0)+d(x0,y)<r+ d(x0,y)=ry: hence A⊆B(y,ry).

Example 4.4 Show that union of a finite number of bounded sets in a MS is bounded. Prove through a counterexample that condition of finiteness is necessary.

**Definition 4.2** Let A be a nonempty subset of a MS (X,d). The diameter of A, diam(A), is defined by : diam(A)=sup{d(x,y): x,yA}, if sup exists; otherwise A is of infinite diameter.

Example 4.5 Show that a subset A is bounded iff either it is empty or its diameter diam(A) is finite.

Proof Let A be bounded. Then there exists xX and r>0 such that AB(x,r).Let A .For a1,a2A, d(a1,a2)d(a1,x)+d(x,a2)<2r; thus diam(A)=sup{d(a,b):a,bA}; thus diam(A) is finite.

Conversely, let diam(A) be finite, say, r>0. Let A and let a0A. For aA, d(a,a0) sup{d(a,b):a,bA}=r; thus, A⊆B(a0,r); hence A is bounded.

Example 4.6 Show that diam(B(x,r))2r and that the strict inequality can occur.

Answer In a discrete MS (X,d), diam(B(x,1/2))= diam{x}=0<2.(1/2)=1.

Example 4.7 Consider R with the standard metric d and the metric (x,y)= min{d(x,y),1}. Prove that (R, is bounded while (R,d) is not.

Answer For xR, (x,0)1. Hence RB(0,1) in (R,; hence R is bounded in (R,. In (R,d), diam(R)=sup{x,yR} does not exist; hence R is not bounded in (R,.

Note The property of ‘boundedness’ is metric specific: there may exist two metrics d1 and d2 on a set X such that X is bounded in (X,d1) whereas not bounded in (X,d2).

**Definition 4.3** Let (X,d) be a MS and let A,B be nonempty subsets of X. Then the **distance** between the sets A and B, denoted by d(A,B), is defined by d(A,B)= inf{d(a,b): aA, b}. If ,say, A is a singleton {a}, then we shall denote d({a},B) by d(a,B).

Note Since in a MS (X,d), d(x,y)0 for all x,yX, so d(A,B) always exists and is non-negative. If , then d(A,B)=0; but the converse need not hold: consider A=(0,1),B=(1,2) in Ru.

**Theorem 4.1**  Let (X,d) be a MS and A,B⊆X. Then (1) A⊆B⟹ diam(A)diam(B), (2) x,yX⟹ (3) diam(AB)diam(A)+diam(B) (4) diam(A)= diam()

Proof (2) If d(x,A)=d(y,A), nothing remains to be proved. For definiteness sake, let d(x,A)>d(y,A). Let >0 be arbitrary. Since d(y,A)=inf{d(y,a):aA}, there exists aA such that d(y,a)<d(y,A)+ Then d(x,A)-d(y,A)<d(x,A)-d(y,a)+d(x,a)-d(y,a)+ (since d(x,A)d(x,a)) d(x,y)+, which gives , since is arbitrary.

(4) Clearly d(A)). Let x,y. Let >0 be given. There exists a1,a2A such that d(x,a1)</2 and d(y,a­2)</2. Thus d(x,y)d(x,a1)+d(a1,a2)+d(a2,y)<d(a1,a2)+ d(A)+. Hence ,d( )=sup{d(x,y):x} d(A)+ .By arbitrariness of , ) d(A); combining diam(A)= diam().

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