Module-II [Elementary Operation Research (Optional Module)] (36 classes) (50 marks)

Motivation of Linear Programming Problem. Statement and formulation of L.P.P. Solution by graphical method (for two variables),

Convex set, hyperplane, extreme points, convex polyhedron, basic solutions and basic feasible solutions (b.f.s.). Degenerate and non-degenerate b.f.s..

The set of all feasible solutions of an L.P.P.is a convex set. The objective function of an L.P.P. assumes its optimal value at an extreme point of the convex set of feasible solutions. A b.f.s. to an L.P.P. corresponds to an extreme point of the convex set of all feasible solutions.

Fundamental Theorem of L.P.P.(statement only). Reduction of a feasible solution to a b.f.s.

Standard form of an L.P.P. Solution by simplex method and method of penalty .

Duality theory-The dual of the dual is the primal, relation between the objective values of dual and the primal problems. Dual problems with at most one unrestricted variable and one constraint of equality.

Transportation and Assignment problem and their optimal solutions.

Inventory Control.

LECTURE NOTES ON LINEAR PROGRAMMING

Pre-requisites: Matrices and Vectors

CHAPTER I

Mathematical formulation of Linear Programming Problem

Let us consider two real life situations to understand what we mean by a programming problem. For any industry, the objective is to earn maximum profit by selling products which are produced with limited available resources, keeping the cost of production at a minimum. For a housewife the aim is to buy provisions for the family at a minimum cost which will satisfy the needs of the family.

All these type of problems can be done mathematically by formulating a problem which is known as a programming problem. Some restrictions or constraints are to be adopted to formulate the problem. The function which is to be maximized or minimized is called the objective function. If in a programming problem the constraints and the objective function are of linear type then the problem is called a linear programming problem. There are various types of linear programming problems which we will consider through some examples.

Examples

1. (Production allocation problem) Four different type of metals, namely, iron, copper, zinc and manganese are required to produce commodities A, B and C. To produce one unit of A, 40kg iron, 30kg copper, 7kg zinc and 4kg manganese are needed. Similarly, to produce one unit of B, 70kg iron, 14kg copper and 9kg manganese are needed and for producing one unit of C, 50kg iron, 18kg copper and 8kg zinc are required. The total available quantities of metals are 1 metric ton iron, 5 quintals copper, 2 quintals of zinc and manganese each. The profits are Rs 300, Rs 200 and Rs 100 by selling one unit of A, B and C respectively. Formulate the problem mathematically.
   Solution: Let z be the total profit and the problem is to maximize z(called the objective function). We write below the given data in a tabular form:

	Iron	Copper	Zinc	Manganese	Profit per unit in Rs
A	40kg	30kg	7kg	4kg	300
B	70kg	14kg	0kg	9kg	200
C	60kg	18kg	8kg	0kg	100
Available quantities→	1000kg	500kg	200kg	200kg	

To get maximum profit, suppose x_1 units of A, x_2 units of B and x_3 units of C are to be produced. Then the total quantity of iron needed is $(40x_1 + 70x_2 + 60x_3)$kg. Similarly, the total quantity of copper, zinc and manganese needed are $(30x_1 + 14x_2 + 18x_3)$kg , $(7x_1 + 0x_2 + 8x_3)$kg and $(4x_1 + 9x_2 + 0x_3)$kg respectively. From the conditions of the problem we have,

$$40x_1 + 70x_2 + 60x_3 \leq 1000$$

$$30x_1 + 14x_2 + 18x_3 \leq 500$$

$$7x_1 + \ 0x_2 + \ \ 8x_3 \leq 200$$

$$4x_1 + \ 9x_2 + \ 0x_3 \leq 200$$

The objective function is $z = 300x_1 + 200x_2 + 100x_3$ which is to be maximized. Hence the problem can be formulated as,

Maximize $z = 300x_1 + 200x_2 + 100x_3$

Subject to $40x_1 + 70x_2 + 60x_3 \leq 1000$

$$30x_1 + 14x_2 + 18x_3 \leq 500$$

$$7x_1 + \ 0x_2 + \ \ 8x_3 \leq 200$$

$$4x_1 + \ 9x_2 + \ 0x_3 \leq 200$$

As none of the commodities produced can be negative, $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$.

All these inequalities are known as constraints or restrictions.

2. (Diet problem) A patient needs daily 5mg, 20mg and 15mg of vitamins A, B and C respectively. The vitamins available from a mango are 0.5mg of A, 1mg of B, 1mg of C, that from an orange is 2mg of B, 3mg of C and that from an apple is 0.5mg of A, 3mg of B, 1mg of C. If the cost of a mango, an orange and an apple be Rs 0.50, Rs 0.25 and Rs 0.40 respectively, find the minimum cost of buying the fruits so that the daily requirement of the patient be met. Formulate the problem mathematically.

Solution: The problem is to find the minimum cost of buying the fruits. Let z be the objective function. Let the number of mangoes, oranges and apples to be bought so that the cost is minimum and to get the minimum daily requirement of the vitamins be x_1, x_2, x_3 respectively. Then the objective function is given by

$$z = 0.50\ x_1 + 0.25\ x_2 + 0.40\ x_3$$

From the conditions of the problem

$0.5x_1 + 0x_2 + 0.5x_3 \geq 5$
$x_1 + 2x_2 + \quad 3x_3 \geq 2$
$x_1 + \ 3x_2 + \quad x_3 \geq 15$ and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

Hence the problem is

Minimize $z = 0.50\, x_1 + 0.25\, x_2 + 0.40\, x_3$.

Subject to $0.5x_1 + 0x_2 + 0.5x_3 \geq 5$

$x_1 + 2x_2 + \quad 3x_3 \geq 20$

$x_1 + \ 3x_2 + \quad x_3 \geq 15$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

3. (Transportation problem) Three different types of vehicles A, B and C have been used to transport 60 tons of solid and 35 tons of liquid substance. Type A vehicle can carry 7 tons solid and 3 tons liquid whereas B and C can carry 6 tons solid and 2 tons liquid and 3 tons solid and 4 tons liquid respectively. The cost of transporting are Rs 500, Rs 400 and Rs 450 respectively per vehicle of type A, B and C respectively. Find the minimum cost of transportation. Formulate the problem mathematically.
   Solution: Let z be the objective function. Let the number of vehicles of type A, B and C used to transport the materials so that the cost is minimum be x_1, x_2, x_3 respectively. Then the objective function is $= 500x_1 + 400x_2 + 450x_3$. The quantities of solid and liquid transported by the vehicles are $7x_1 + 6x_2 + 3x_3$ tons and $3x_1 + 2x_2 + 4x_3$ tons respectively. By the conditions of the problem, $7x_1 + 6x_2 + 3x_3 \geq 60$ and $3x_1 + 2x_2 + 4x_3 \geq 35$. Hence the problem is
   Minimize $z = 500x_1 + 400x_2 + 450x_3$
   Subject to $7x_1 + 6x_2 + 3x_3 \geq 60$
   $3x_1 + 2x_2 + 4x_3 \geq 35$

   And $x_1, x_2, x_3 \geq 0$

4. An electronic company manufactures two radio models each on a separate production line. The daily capacity of the first line is 60 radios and that of

the second line is 75 radios. Each unit of the first model uses 10 pieces of a certain electronic component, whereas each unit of the second model uses 8 pieces of the same component. The maximum daily availability of the special component is 800 pieces. The profit per unit of models 1 and 2 are Rs 500 and Rs 400 respectively. Determine the optimal daily production of each model.

Solution: This is a maximization problem. Let x_1, x_2 be the number of two radio models each on a separate production line. Therefore the objective function is $z = 500x_1 + 400x_2$ which is to be maximized. From the conditions of the problem we have $x_1 \leq 60$, $x_2 \leq 75$, $10x_1 + 8x_2 \leq 800$. Hence the problem is

Maximize $z = 500x_1 + 400x_2$
Subject to $x_1 \leq 60$
$x_2 \leq 75$
$10x_1 + 8x_2 \leq 800$
And $x_1, x_2 \geq 0$

5. An agricultural firm has 180 tons of Nitrogen fertilizers, 50 tons of Phosphate and 220 tons of Potash. It will be able to sell 3:3:4 mixtures of these substances at a profit of Rs 15 per ton and 2:4:2 mixtures at a profit of Rs 12 per ton respectively. Formulate a linear programming problem to determine how many tons of these two mixtures should be prepares so as to maximize profit.

   Solution: Let the 3:3:4 mixture be called A and 2:4:2 mixture be called B. Let x_1 , x_2 tons of these two mixtures be produced to get maximum profit. Thus the objective function is $z = 15x_1 + 12x_2$ which is to be maximized. Let us denote Nitrogen, Phosphate and Potash as N Ph and P respectively.

   Then in the mixture A , $\frac{N}{3} = \frac{Ph}{3} = \frac{P}{4} = k_1$(say).

   $$\Rightarrow N = 3k_1 , Ph = 3k_1 , P = 4k_1$$

   $$\Rightarrow x_1 = 10k_1 .$$

   Similarly for the mixture B , $N = 2k_2 , Ph = 4k_2 , P = 2k_2k_1$

   $$\Rightarrow x_2 = 8k_2 .$$

Thus the constraints are $\frac{3}{10}x_1 + \frac{1}{4}x_2 \leq 180$ [since in A, the amount of nitrogen is $\frac{3k_1}{10k_1}x_1 = \frac{3}{10}x_1$].

Similarly $\frac{3}{10}x_1 + \frac{1}{2}x_2 \leq 250$ and $\frac{2}{5}x_1 + \frac{1}{4}x_2 \leq 220$.

Hence the problem is

Maximize $z = 15x_1 + 12x_2$

Subject to $\frac{3}{10}x_1 + \frac{1}{4}x_2 \leq 180$

$\frac{3}{10}x_1 + \frac{1}{2}x_2 \leq 250$

$\frac{2}{5}x_1 + \frac{1}{4}x_2 \leq 220$

And $x_1, x_2 \geq 0.$

6. A coin to be minted contains 40% silver, 50% copper, 10% nickel. The mint has available alloys A, B, C and D having the following composition and costs, and availability of alloys:

	% silver	% copper	% nickel	Costs per Kg
A	30	60	10	Rs 11
B	35	35	30	Rs 12
C	50	50	0	Rs 16
D	40	45	15	Rs 14
Availability of alloys →	Total 1000 Kgs			

Present the problem of getting the alloys with specific composition at minimum cost in the form of a L.P.P.

Solution: Let x_1, x_2, x_3, x_4 Kg s be the quantities of alloys A, B, C, D used for the purpose. By the given condition $x_1 + x_2 + x_3 + x_4 \leq 1000$.

The objective function is $z = 11x_1 + 12x_2 + 16x_3 + 14x_4$

and the constraints are $0.3x_1 + 0.35x_2 + 0.5x_3 + 0.4x_4 \geq 400$ for silver

$0.6x_1 + 0.35x_2 + 0.5x_3 + 0.45x_4 \geq 500$ for copper

$$0.1x_1 + \quad 0.3x_2 + \qquad + 0.15x_4 \geq 100$$ for nickel

Thus the L.P.P is Minimize $z = 11x_1 + 12x_2 + 16x_3 + 14x_4$

Subject to

$$0.3x_1 + 0.35x_2 + 0.5x_3 + \ 0.4x_4 \geq 400$$
$$0.6x_1 + 0.35x_2 + 0.5x_3 + 0.45x_4 \geq 500$$
$$0.1x_1 + \ 0.3x_2 + \qquad + 0.15x_4 \geq 100$$
$$x_1 + \quad x_2 + \quad x_3 + \quad x_4 \leq 1000$$

And $x_1, x_2, x_3 \geq 0$

7. A hospital has the following minimum requirement for nurses.

Period	Clock time (24 hours day)	Minimum number of nurses required
1	6A.M-10A.M	60
2	10A.M-2P.M	70
3	2P.M-6P.M	60
4	6P.M-10P.M	50
5	10P.M-2A.M	20
6	2A.M-6A.M	30

Nurses report to the hospital wards at the beginning of each period and work for eight consecutive hours. The hospital wants to determine the minimum number of nurses so that there may be sufficient number of nurses available for each period. Formulate this as a L.P.P.

Solution: This is a minimization problem. Let $x_1, x_2, \ldots\ldots, x_6$ be the number of nurses required for the period 1, 2, ……, 6. Then the objective function is

Minimize, $z = x_1 + x_2 + \cdots \ldots + x_6$ and the constraints can be written in the following manner.

x_1 nurses work for the period 1 and 2 and x_2 nurses work for the period 2 and 3 etc. Thus for the period 2,

$x_1 + x_2 \geq 70.$

Similarly, for the periods 3, 4, 5, 6, 1 we have,

$x_2 + x_3 \geq 60$

$x_3 + x_4 \geq 50$

$x_4 + x_5 \geq 20$

$x_5 + x_6 \geq 30$

$x_6 + x_1 \geq 60$, $x_j \geq 0, j = 1, 2, \ldots\ldots, 6$

Mathematical formulation of a L.P.P.

From the discussion above, now we can mathematically formulate a general Linear Programming Problem which can be stated as follows.

Find out a set of values $x_1, x_2, \ldots\ldots, x_n$ which will optimize (either maximize or minimize) the linear function

$z = c_1x_1 + c_2x_2 + \cdots \ldots + c_nx_n$

Subject to the restrictions

$a_{11}x_1 + a_{12}x_2 + \cdots \ldots + a_{1n}x_n (\leq = \geq) b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots \ldots + a_{2n}x_n (\leq = \geq) b_2$

……………………………………………..

$a_{m1}x_1 + a_{m2}x_2 + \cdots \ldots + a_{mn}x_n (\leq = \geq) b_m$

And the non-negative restrictions $x_j \geq 0, j = 1, 2, \ldots\ldots, n$ where a_{ij}, c_j, b_i $(i = 1, 2, \ldots\ldots, m,\ j = 1, 2, \ldots\ldots, n)$ are all constants and x_j , $(j = 1, 2, \ldots\ldots, n)$ are variables. Each of the linear expressions on the left hand side connected to the corresponding constants on the right side by only one of the signs ≤ , = and ≥ ,is known as a constraint. A constraint is either an equation or an inequation.

The linear function $z = c_1x_1 + c_2x_2 + \cdots \ldots + c_nx_n$ is known as the objective function.

By using the matrix and vector notation the problem can be expressed in a compact form as

Optimize $z = c^T x$ subject to the restrictions $Ax \leq = \geq b, x \geq 0$,

where $A = [a_{ij}]$ is a m x n coefficient matrix.,

$c = (c_1, c_2, \ldots \ldots, c_n)^T$ is a n-component column vector, which is known as a cost or price vector,

$x = (x_1, x_2, \ldots \ldots, x_n)^T$ is a n-component column vector, which is known as decision variable vector or legitimate variable vector and

$b = (b_1, b_2, \ldots \ldots, b_m)^T$ is a m-component column vector, which is known as requirement vector.

In all practical discussions, $b_i \geq 0 \; \forall i.$ If some of them are negative, we make them by multiplying both sides of the inequality by (-1).

If all the constraints are equalities, then the L.P.P is reduced to

Optimize $z = c^T x$ subject to $Ax = b, x \geq 0$.

This form is called the standard form.

Feasible solution to a L.P.P: A set of values of the variables, which satisfy all the constraints and all the non-negative restrictions of the variables, is known as the feasible solution (F.S.) to the L.P.P.

Optimal solution to a L.P.P: A feasible solution to a L.P.P which makes the objective function optimal is known as the optimal solution to the L.P.P

There are two ways of solving a linear programming problem: (1) Geometrical method and (2) Algebraic method.

A particular L.P.P is either a minimization or a maximization problem. The problem of minimization of the objective function z is nothing but the problem of maximization of the function $(-z)$ and vice versa and $\min z = -\max(-z)$ with the same set of constraints and the same solution set.

Graphical or Geometrical Method of Solving a Linear Programming Problem

We will illustrate the method by giving examples.

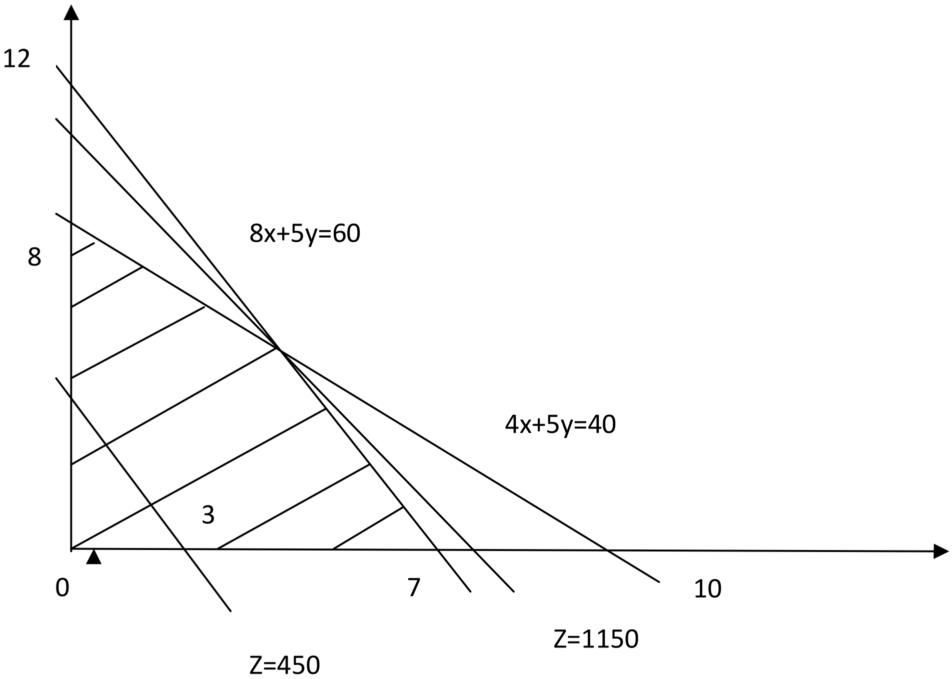
Examples

Solve the following problems graphically.

1. Maximize $z = 150x + 100y$

   Subject to $8x + 5y \leq 60$

   $4x + 5y \leq 40$, $x, y \geq 0$.

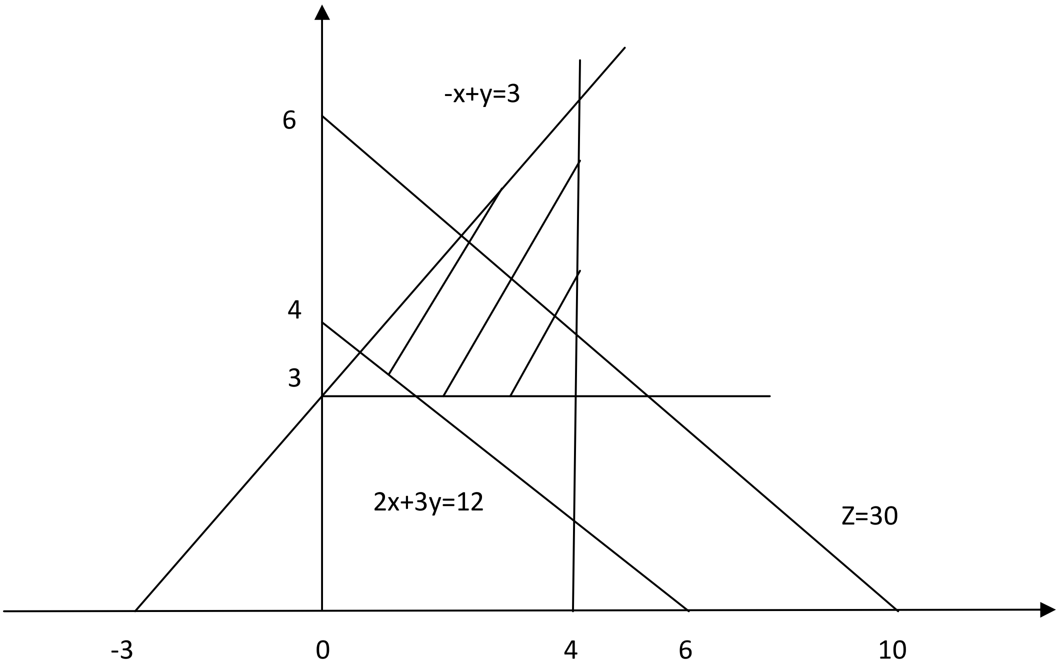


The constraints are treated as equations along with the non negativity relation. We confine ourselves to the first quadrant of the xy plane and draw the lines given by those equations. Then the directions of the inequalities indicate that the striped region in the graph is the feasible region. For any particular value of z, the graph of the objective function regarded as an equation is a straight line (called the profit line in a maximization problem) and as z varies, a family of parallel lines is generated. We have drawn the line corresponding to z=450. We see that the profit z is proportional to the perpendicular distance of this straight line from the origin.

Hence the profit increases as this line moves away from the origin. Our aim is to find a point in the feasible region which will give the maximum value of z. In order to find that point we move the profit line away from origin keeping it parallel to itself. By doing this we find that (5,4) is the last point in the feasible region which the moving line encounters. Hence we get the optimal solution $z_{max} = 1150$ for $= 5, y = 4$.

Note: If we have a function to minimize, then the line corresponding to a particular value of the objective function (called the cost line in a minimization problem) is moved towards the origin.

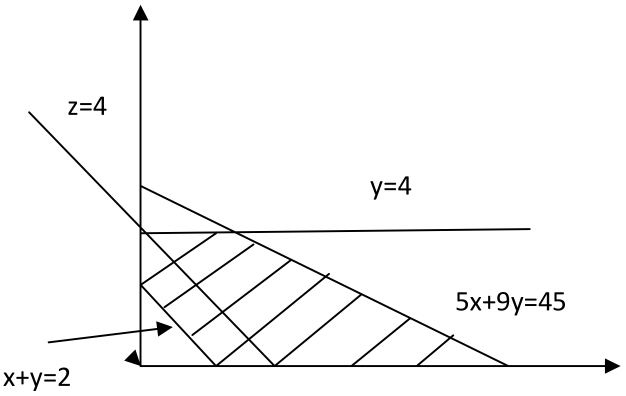
2. Solve graphically:
   Minimize $z = 3x + 5y$
   Subject to $2x + 3y \geq 12$
   $-x + y \leq 3$
   $x \leq 4$
   $y \leq 3$



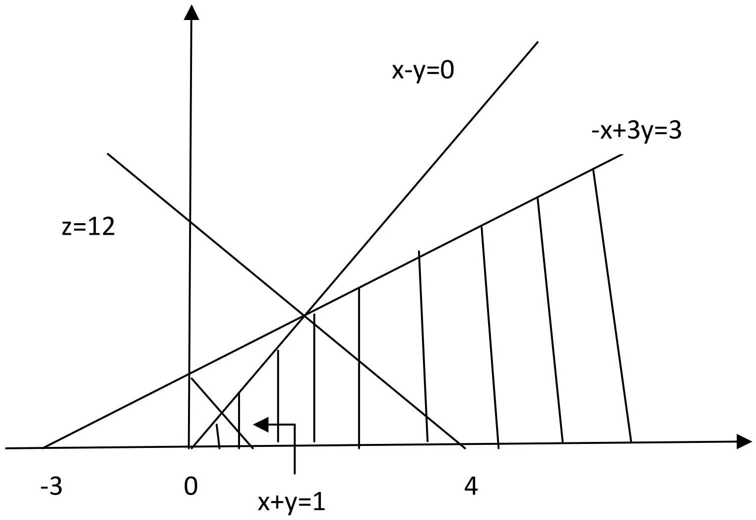
Here the striped area is the feasible region. We have drawn the cost line corresponding to z=30. As this is a minimization problem the cost line is moved towards the origin and the cost function takes its minimum at $z_{min} = 19.5$ for $= 1.5, y = 3$.

In both the problems above the L.P.P. has a unique solution.

3. Solve graphically:
   Minimize $z = x + y$
   Subject to $5x + 9y \leq 45$
   $x + y \geq 2$
   $y \leq 4 \quad , x, y \geq 0$
   Here the striped area is the feasible region. We have drawn the cost line corresponding to z=4. As this is a minimization problem the cost line when moved towards the origin coincides with the boundary line $x + y = 2$ and the optimum value is attained at all points lying on the line segment joining (2,0) and (0,2) including the end points. Hence there are an infinite number of solutions. In this case we say that alternative optimal solution exists.



4.
5.
6. Solve graphically
   Maximize $z = 3x + 4y$
   Subject to $x - y \geq 0$
   $x + y \geq 1$
   $-x + 3y \leq 3 , x, y \geq 0$



The striped region in the graph is the feasible region which is unbounded.. For any particular value of z, the graph of the objective function regarded as an equation is a straight line (called the profit line in a maximization problem) and as z varies, a family of parallel lines is generated. We have drawn the line corresponding to z=12. We see that the profit z is proportional to the perpendicular distance of this straight line from the origin. Hence the profit increases as this line moves away from the origin. As we move the profit line away from origin keeping it parallel to itself we see that there is no finite maximum value of z.

Ex: Keeping everything else unaltered try solving the problem as a minimization problem.

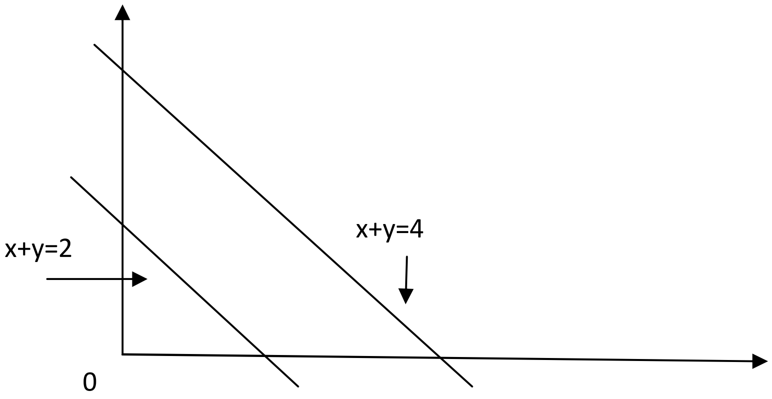
7. Solve graphically

   Maximize $z = 2x - 3y$

   Subject to $x + \ y \leq \ 2$

   $x + \ y \geq 4$

   $x, y \ \geq 0$



It is clear that there is no feasible region.

In algebraic method, the problem can be solved only when all constraints are equations. We now show how the constraints can be converted into equations.

Slack and Surplus Variables

When the constraints are inequations connected by the sign " $\leq$ " , in each inequation a variable is added on the left hand side of it to convert ind sidet into an equation. For example, the constraint

$$x_1 - 2x_2 + 7x_3 \leq 4$$

is connected by the sign $\leq$. Then a variable x_4 is added to the left hand side and it is converted into an equation

$$x_1 - 2x_2 + 7x_3 + x_4 = 4$$

From the above it is clear that the slack variables are non-negative quantities.

If the constraints are connected by " $\geq$ " ,in each inequation a variable is subtracted from the left hand side to convert it into an equation. These variables are known as surplus variables. For example,

$$x_1 - 2x_2 + 7x_3 \geq 4$$

is converted into an equation by subtracting a variable x_4 from the left hand side.

$$x_1 - 2x_2 + 7x_3 - x_4 = 4$$

The surplus variables are also non-negative quantities.

Let a general L.P.P containing r variables and m constraints be

Optimize $z = c_1x_1 + c_2x_2 + \cdots \ldots + c_rx_r$

subject to $a_{i1}x_1 + a_{i2}x_2 + \cdots \ldots + a_{in}x_n \leq = \geq b_i ,\ i = 1,2, \ldots \ldots , m,\ x_j \geq 0, j = 1,2, \ldots \ldots , r,$

where one and only one of the signs $\leq, =, \geq$ holds for each constraint, but the signs may vary from one constraint to another. Let k constraints out of the m be inequations ($0 \leq k \leq m$). Then introducing k slack or surplus variables $x_{r+1}, x_{r+2}, \dots \dots, x_n, n = r + k$, one to each of the inequations, all constraints can be converted into equations containing n variables. We further assume that $n \geq m.$ The objective function is similarly accommodated with k slack or surplus variables $x_{r+1}, x_{r+2}, \dots \dots, x_n$, the cost components of these variables are assumed to be zero. Then the adjusted objective function is

$z_{ad} = c_1x_1 + c_2x_2 + \cdots \dots + c_rx_r + 0x_{r+1} + 0x_{r+2} + \cdots \dots + 0x_n$, and then the problem can be written as

Optimize $z_{ad} = c^Tx$ subject to $Ax = b, x \geq 0,$

where A is an mxn matrix , known as coefficient matrix given by

$A = (a_1, a_2, \dots \dots, a_n),$

where $a_j = (a_{1j}, a_{2j}, \dots \dots, a_{mj})^T$ is a column vector associated with the vector x_j, $j = 1,2, \dots \dots, n$.

$c = (c_1, c_2, \dots \dots, c_r, 0,0, \dots ,0)^T$ is a n-component column vector,

$x = (x_1, x_2, \dots \dots, x_r, x_{r+1}, x_{r+2}, \dots \dots, x_n)^T$ is a n-component column vector, and

$b = (b_1, b_2, \dots \dots, b_m)^T$ is a m-component column vector.

The components of b can be made positive by proper adjustments.

It is worth noting that the column vectors associated with the slack variables are all unit vectors. As the cost components of the slack and surplus variables are all zero, it can be verified easily that the solution set which optimizes z_{ad} also optimizes z. Hence to solve the original L.P.P it is sufficient to solve the standard form of the L.P.P. So, for further discussions we shall use the same notation for z_{ad} and z.

Problems

1. Transform the following Linear Programming Problems to the standard form:
   (i) Maximize $z = 2x_1 + 3x_2 - 4x_3$

Subject to $4x_1 + 2x_2 - x_3 \leq 4$

$-3x_1 + 2x_2 + 3x_3 \geq 6$

$x_1 + x_2 - 3x_3 = 8$, $x_j \geq 0, j = 1,2,3$.

Solution: First constraint is $\leq$ type and the second one is a $\geq$ type, so adding a slack and a surplus variable respectively, the two constraints are converted into equations. Hence the transformed problem can be written as

Maximize $z = 2x_1 + 3x_2 - 4x_3 + 0x_4 + 0x_5$

Subject to $4x_1 + 2x_2 - x_3 + x_4 = 4$

$-3x_1 + 2x_2 + 3x_3 - x_5 = 6$

$x_1 + x_2 - 3x_3 = 8$, $x_j \geq 0, j = 1,2,3,4,5$.

(ii) Maximize $z = x_1 - x_2 + x_3$

Subject to $x_1 + x_2 - 3x_3 \geq 4$

$2x_1 - 4x_2 + x_3 \geq -5$

$x_1 + 2x_2 - 2x_3 \leq 3$, $x_j \geq 0, j = 1,2,3$.

Solution: The problem can be transformed as

Maximize $z = x_1 - x_2 + x_3 + 0x_4 + 0x_5 + 0x_6$

Subject to $x_1 + x_2 - 3x_3 + x_4 = 4$

$2x_1 - 4x_2 + x_3 - x_5 = -5$

$x_1 + 2x_2 - 2x_3 - x_6 = 3$,

$x_j \geq 0, j = 1,2,3,4,5,6$.

x_4, x_5 are surplus and x_6 is a slack variable. Making the second component of b vector positive , the second equation can be written as

$-2x_1 + 4x_2 - x_3 + x_5 = 5$

and in that case the surplus variable is changed into a slack variable.

2. Express the following minimization problem as a standard maximization problem by introducing slack and surplus variables.

Minimize $z = 4x_1 - x_2 + 2x_3$

Subject to $4x_1 + x_2 - x_3 \leq 7$

$2x_1 - 3x_2 + x_3 \leq 12$

$x_1 + x_2 + x_3 = 8$

$4x_1 + 7x_2 - x_3 \geq 16$, $x_j \geq 0, j = 1,2,3$.

Solution: After introducing slack variables in the first two constraints and a surplus in the fourth, the converted problem is,

Minimize $z^* = (-z) = 4x_1 - x_2 + 2x_3 + 0x_4 + 0x_5 + 0x_6$

$$
\begin{aligned}
\text{Subject to}\quad & 4x_1 + x_2 - x_3 + x_4 && = 7 \\
& 2x_1 - 3x_2 + x_3 + x_5 && = 12 \\
& x_1 + x_2 + x_3 && = 8 \\
& 4x_1 + 7x_2 - x_3 - x_6 && = 16 \quad , \; x_j \geq 0, j = 1,2, \dots ,6 \; .
\end{aligned}
$$

Writing the above problem as a standard maximization problem

$$
\begin{aligned}
\text{Maximize}\quad & z^* = (-z) = 4x_1 - x_2 + 2x_3 + 0x_4 + 0x_5 + 0x_6 \\
\text{Subject to}\quad & 4x_1 + x_2 - x_3 + x_4 && = 7 \\
& 2x_1 - 3x_2 + x_3 + x_5 && = 12 \\
& x_1 + x_2 + x_3 && = 8 \\
& 4x_1 + 7x_2 - x_3 - x_6 && = 16 \quad , \; x_j \geq 0, j = 1,2, \dots ,6 \; .
\end{aligned}
$$

Variable unrestricted in sign

If a variable x_j is unrestricted in sign, then it can be expressed as a difference of two non-negative variables, say, $x_j^{/}, x_j^{//}$ as $x_j = x_j^{/} - x_j^{//}$, $x_j^{/} \geq 0, x_j^{//} \geq 0$. If $x_j^{/} > x_j^{//}$, then $x_j > 0$, if $x_j^{/} = x_j^{//}$, then $x_j = 0$ and if $x_j^{/} < x_j^{//}$, then $x_j < 0$. Hence x_j is unrestricted in sign.

3. Write down the following L.P.P in the standard form.

Maximize $z = 2x_1 + 3x_2 - x_3$

$$
\begin{aligned}
\text{Subject to}\quad & 4x_1 + x_2 + x_3 \geq 4 \\
& 7x_1 + 4x_2 - x_3 \leq 25 \; , \; x_j \geq 0, j = 1,3 \; , x_2 \text{ unrestricted in sign .}
\end{aligned}
$$

Solution: Introducing slack and surplus variables and writing $x_2 = x_2^{/} - x_2^{//}$, where $x_2^{/} \geq 0, x_2^{//} \geq 0$, the problem in the standard form is

$$\text{Maximize} \quad z = 2x_1 + 3x_2^{/} - 3x_2^{//} - x_3 + 0x_4 + 0x_5$$

$$\begin{aligned} \text{Subject to} \quad 4x_1 + \ \ x_2^{/} - \ \ x_2^{//} + x_3 - x_4 \qquad &= \ 4 \\ 7x_1 + \ 4x_2^{/} - 4\, x_2^{//} - x_3 \qquad + x_5 &= 25 \end{aligned} \qquad ,$$

$x_1 . x_2^{/}, x_2^{//}, x_3 \geq 0$.

CHAPTER II

Basic Solutions of a set of Linear Simultaneous Equations

Let us consider m linear equations with n variables $(n > m)$ and let the set of equations be

$$a_{11}x_1 + a_{12}x_2 + \cdots \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots \ldots + a_{2n}x_n = b_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots \ldots + a_{mn}x_n = b_m$$

This set of equations can be written in a compact form as

$Ax = b,$ where,

$A = [a_{ij}]$ is the coefficient matrix of order m x n,

$x = (x_1, x_2, \ldots \ldots, x_n)^T$ is a n-component column vector,

$b = (b_1, b_2, \ldots \ldots, b_m)^T$ is a m-component column vector.

We further assume that $R(A) = m,$ which indicates that all equations are linearly independent and none of them are redundant.

The set of equations can also be written in the form

$x_1a_1 + x_2a_2 + \cdots \ldots. + x_ja_j + \cdots \ldots + x_na_n = b$ where $a_j = (a_{1j}, a_{2j}, \ldots \ldots, a_{mj})^T$, an m component column vector and all are non-null vectors. These vectors are called activity vectors. From the theory of linear algebra, we know that here infinitely many solutions exist. We will now find a particular type of solutions of the set of equations which are finite in number.

From the set of n column vectors a_j we arbitrarily select m linearly independent vectors (there exists at least one such set of vectors since $(A) = m$, and $n > m$) which constitutes a basis B of the Euclidean space R^m. The vectors which are not included in the selected set are called non-basic vectors. Assuming that all variables associated with the non-basic vectors are zero, we get a set of m equations with m variables. The coefficient matrix B here is the basis matrix and hence is non-singular. Hence there exists a unique solution for the set of m equations containing m variables. This solution is called a Basic Solution. The variables associated with the basis vectors are called basic variables. The number of basic variables are m, and the number of non-basic variables(the ones associated with the non-basic vectors) are $n - m$ whose values are assumed to be zero. Then the set of equations are reduced to

$Bx_B = b,$

Where B is the basis matrix and x_B is the m component column vector consisting of the basic variables. Using the matrix inversion method of finding the solution of a set of equations

$(B^{-1}B)x_B = B^{-1}b,$ or, $I_mx_B = x_B = B^{-1}b$, where x_B is the m-component column vector written as $x_B = (x_{B1}, x_{B2}, \ldots\ldots, x_{Bm})$.

The general solution is written as $x_B = [B^{-1}b, 0]^T$, where 0 is a $(n - m)$ component null vector.

Since out of n vectors, m vectors constitute a basis, then theoretically the maximum number of basis matrices are ^nC_m and hence the maximum number of basic solutions are ^nC_m . Hence the basic solutions are finite in number. We now formally define a basic solution.

Basic Solution: Let us consider a system of m simultaneous linear equations containing n variables $(n > m)$ and write the set of equations as $Ax = b$, where $R(A) = m$. If any mxm arbitrary non-singular sub-matrix(say B), be selected from A, and we assume all $(n - m)$ variables not associated with the column vectors of B are zero, then the solution so obtained is called a basic solution. The m variables associated with the columns of the non-singular matrix B are called basic variables and the remaining $n - m$ variables whose values are assumed to be zero, are called non-basic variables. The values of each of the basic variables can be positive, negative or zero. From this we can conclude that a solution is said to be a basic solution if the vectors a_j associated with the non-zero vectors are linearly independent. This condition is necessary and sufficient.

Non-Degenerate Basic Solution: If the values of all the basic variables are non-zero then the basic solution is known as a Non-Degenerate Basic Solution.

Degenerate Basic Solution: If the value of at least one basic variable is zero then the basic solution is known as a Non-Degenerate Basic Solution.

Basic Feasible Solution (B.F.S): The solution set of a L.P.P. which is feasible as well as basic is known as a Basic Feasible Solution.

Non-degenerate B.F.S: The solution to a L.P.P. where all the components corresponding to the basic variables are positive is called a Non-degenerate B.F.S.

Degenerate B.F.S: The solution to a L.P.P. where the value of at least one basic variable is zero is called a Degenerate B.F.S.

Examples

1. Find the basic solutions of the system of equations given below and identify the nature of the solution.
   $$2x_1 + 4x_2 - 2x_3 = 10$$
   $$10x_1 + 3x_2 + 7x_3 = 33$$
2. Given that $x_1 = 2,\ x_2 = -1,\ x_3 = 0$ is a solution of a system of equations
   $$3x_1 - 2x_2 + \ \ x_3 = \ \ 8$$
   $$9x_1 - 6x_2 + 4x_3 = 24$$
   Is this solution basic? Justify.

CHAPTER III

N-Dimensional Euclidean Space and Convex Set

We will denote the N-Dimensional Euclidean Space by V_n or E^n or R^n . The points in E^n are all column vectors.

Point Set: Point sets are sets whose elements are all points in E^n .

Line: If $x_1 = (x_{11}, x_{12}, \dots \dots, x_{1n})^T$ and $x_2 = (x_{21}, x_{22}, \dots \dots, x_{2n})^T$ be two points in E^n , then the line joining the points x_1 and x_2 , $(x_1 \neq x_2)$ is a set X of points given by

$$X = \{x: x = \lambda x_1 + (1 - \lambda)x_2, for\ all\ real\ \lambda\}$$

Line segment: If $x_1 = (x_{11}, x_{12}, \dots \dots, x_{1n})^T$ and $x_2 = (x_{21}, x_{22}, \dots \dots, x_{2n})^T$ be two points in E^n , then the line segment joining the points x_1 and x_2 , $(x_1 \neq x_2)$ is a set X of points given by

$$X = \{x: x = \lambda x_1 + (1 - \lambda)x_2, 0 \leq \lambda \leq 1, \lambda \epsilon R\}$$

Hyperplane: A hyperplane in E^n is a set X of points given by

$$X = \{x: c^T x = k\} ,$$

Where $c^T = (c_1, c_2, \dots \dots, c_n)$, not all $c_j = 0$, is a fixed element of E^n and $x = (x_1, x_2, \dots \dots, x_n)^T$ is an element of E^n .

A hyperplane can be defined as a set of points which will satisfy $c_1 x_1 + c_2 x_2 + \cdots \dots + c_n x_n = k.$

A hyperplane divides the space E^n into three mutually exclusive disjoint sets given by $x = (x_1, x_2, \dots \dots, x_n)^T$

$$X_1 = \{x: c^T x > k\} , X_2 = \{x: c^T x = k\} , X_3 = \{x: c^T x < k\} .$$

The sets X_1 and X_2 are called open half spaces.

In a L.P.P. , the objective function and the constraints with equality sign are all hyperplanes.

Hypersphere: A hypersphere in E^n with centre at $a = (a_1, a_2, \dots \dots, a_n)^T$ and radius $\varepsilon > 0$ is defined to be the set X of points given by $X = \{x: |x - a| = \varepsilon\}$, where $= (x_1, x_2, \dots \dots, x_n)^T$.

The equation can be written as

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots \dots + (x_n - a_n)^2 = \varepsilon^2$$

The hypersphere in two dimensions is a circle and in three dimensions is a sphere.

An ε- neighbourhood: An ε- neighbourhood about a point a is defined as the set X of points lying inside the hypersphere with centre at a and radius $\varepsilon > 0$, i.e. , the ε-neighbourhood about the point a is a set of points given by $X = \{x: |x - a| < \varepsilon\}$.

An interior point of a set: A point a is an interior point of the set S if there exists an ε- neighbourhood about the point a which contains only points of the same set. From the definition it is clear that an interior point of a set S is an element of the set S.

Boundary point of a set: A point a is a boundary point of a set S if every ε-neighbourhood about the point $a(\varepsilon > 0)$ contains points which are in the set S and points which are not in the set S. A boundary point may or may not be an element of the set .

Open set: A set S is said to be open if it contains only interior points.

Closed set: A set S is said to be closed if it contains all its boundary points.

Bounded set: A set S is said to be strictly bounded set if there exists a positive number r such that for any point x belonging to S, $|x| \leq r$. For every x belonging to S, if $x \geq r$, then the set is bounded from below.

Convex Combination and Convex Sets

Convex Combination of a set of points: The convex combination of a set of k points $x_1, x_2, \dots \dots, x_k$ in a space E^n is also a point x in the same space, given by

$x = \lambda_1 x_1 + \lambda_2 x_2 + \cdots \dots + \lambda_k x_k$ where $\lambda_i \geq 0$ and $\in R$ for all i and $\sum_{i=1}^{k} \lambda_i = 1$.

For different values of the scalar quantities $\lambda_i, i = 1,2, \ldots \ldots, k$ satisfying $\sum_{i=1}^{k} \lambda_i = 1$, and $\lambda_i \geq 0$ for all i, a set of points will be obtained from the convex combinations of the set of k finite points which is a point set in E^n. This point set is known as a convex polyhedron.

The point set X, called the convex polyhedron is given by

$X = \{x : x = \sum_{i=1}^{k} \lambda_i x_i, \sum_{i=1}^{k} \lambda_i = 1$ and $\lambda_i \geq 0$ for all $i\}$.

From the above definition it is also clear that a line segment is a convex combination of two distinct points in the same vector space.

Convex Set: A point set is said to be a convex set if the convex combination of any two points of the set is in the set. In other words, if the line segment joining any two distinct points of the set is in the set then the set is known as a convex set.

Extreme points of a convex set: A point x is an extreme point of the convex set C if it cannot be expressed as a convex combination of two other distinct points x_1, x_2 of the set C, i.e, x cannot be expressed as

$x = \lambda x_1 + (1 - \lambda) x_2, 0 < \lambda < 1$.

From the definition, it is clear that all extreme points of a convex set are boundary points but all boundary points are not necessarily extreme points. Every point of the boundary of a circle is an extreme point of the convex set which includes the boundary and interior of the circle. The extreme points of a square are its four vertices.

Convex hull: If X be a point set, then Convex hull of X which is denoted by $C(X)$, is the set of all convex combinations of set of points from X. If the set X consists of a finite number of points then the convex hull $C(X)$ is called a convex polyhedron.

For a convex polyhedron, any point in the set can be expressed as a convex combination of its extreme points.

Simplex: A simplex is an n-dimensional convex polyhedron having exactly $n + 1$ vertices.

Theorem 1: Intersection of two convex sets is also a convex set.

Proof: Let X_1, X_2 be two convex sets and let $X = X_1 \cap X_2$. It is required to prove that X is also a convex set.

Let x_1, x_2 be two distinct points of X. Then $x_1, x_2 \in X_1$ and $x_1, x_2 \in X_2$. Let x_3 be a point given by

$x_3 = \lambda x_1 + (1 - \lambda) x_2, 0 \leq \lambda \leq 1$.

As x_3 is a convex combination of the points x_1, x_2 and X_1, X_2 are convex sets, then x_3 is a point of X_1 as well as X_2. Hence x_3 is a point of $X = X_1 \cap X_2$. Hence X is a convex set.

Note1: Intersection of a finite number of convex sets is a convex set.

Note 2: Union of two convex sets may not be a convex set.

Theorem 2: A hyperplane is a convex set.

Proof: Let the point set X be a hyperplane given by $X = \{x : c^T x = k\}$. Let x_1, x_2 be two distinct points of X. Then $c^T x_1 = k$ and $c^T x_2 = k$. Let x_3 be a point given by $x_3 = \lambda x_1 + (1 - \lambda) x_2, 0 \leq \lambda \leq 1$.

Therefore, $c^T x_3 = \lambda c^T x_1 + (1 - \lambda) c^T x_2 = \lambda k + (1 - \lambda) k = k$ which indicates that x_3 is also a point of $c^T x = k$.

But x_3 is a convex combination of two distinct points x_1 and x_2 of X. Hence X is a convex set.

Note: Set X is also a closed set.

Theorem 3: Convex polyhedron is a convex set.

Proof: Let S be a point set consisting of a finite number of points $x_1, x_2, \dots \dots, x_k$ in R^n.

We have to show that the convex polyhedron $C(S) = X = \left\{x : x = \sum_{i=1}^{k} \lambda_i x_i, \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1\right\}$.

Let u, v be any two distinct points of X given by

$u = \sum_{i=1}^{k} a_i x_i, a_i \geq 0, \sum_{i=1}^{k} a_i = 1$

$v = \sum_{i=1}^{k} b_i x_i, b_i \geq 0, \sum_{i=1}^{k} b_i = 1$

Consider $w = \lambda u + (1-\lambda)v, 0 \leq \lambda \leq 1$.

Then

$w = \lambda \sum_{i=1}^{k} a_i x_i + (1-\lambda) \sum_{i=1}^{k} b_i x_i = \sum_{i=1}^{k} \{\lambda a_i + (1-\lambda) b_i\} x_i = \sum_{i=1}^{k} c_i x_i$ where $c_i = \lambda a_i + (1-\lambda) b_i$.

Now, $\sum_{i=1}^{k} c_i = \lambda \sum_{i=1}^{k} a_i + (1-\lambda) \sum_{i=1}^{k} b_i$=1 and $c_i \geq 0$ as $a_i \geq 0$, $b_i \geq 0$ and $0 \leq \lambda \leq 1$.

Hence w is also a point of X which is a convex combination of two distinct points of X. Hence X is a convex set.

Theorem 4: The set of all feasible solutions to a L.P.P $Ax = b, x \geq 0$ is a closed convex set.

Proof: Let X be the point set of all feasible solutions of $Ax = b, x \geq 0$.

If the set X has only one point, then there is nothing to prove.

If X has at least two distinct points x_1 and x_2 , then

$Ax_1 = b, x_1 \geq 0$ and $Ax_2 = b, x_2 \geq 0$.

Consider a point x_3 such that $x_3 = \lambda x_1 + (1-\lambda) x_2$, $0 \leq \lambda \leq 1$.

Thus $Ax_3 = \lambda A x_1 + (1-\lambda) A x_2 = \lambda b + (1-\lambda) b = b$.

Again $x_3 \geq 0$ as $x_1 \geq 0$ and $x_2 \geq 0$ and $0 \leq \lambda \leq 1$.

Then x_3 is also a feasible solution to the problem $Ax = b, x \geq 0$.

But x_3 is a convex combination of two distinct points x_1 and x_2 of the set X. Thus X is a convex set.

Now the finite number of constraints represented by $Ax = b$ are closed sets and also the set of inequations (finite) represented by $x \geq 0$ are closed sets and therefore the intersection of a finite number of closed sets which is the set of all feasible solutions is a closed set.

Note: If the L.P.P has at least two feasible solutions then it has an infinite number of feasible solutions

Theorem 5; All B.F.S of the set of equations $Ax = b, x \geq 0$ are extreme points of the convex set of feasible solutions of the equations and conversely.

Proof: Let $A = (a_1, a_2, \dots \dots, a_n)$ be the coefficient matrix of order $m\, x\, n,\ n > m$ and let us assume that B be the basis matrix $B = (a_1, a_2, \dots \dots, a_m)$ where $a_1, a_2, \dots \dots, a_m$ are the column vectors corresponding to the first m variables $x_1, x_2, \dots \dots, x_m$.

Let x be a B.F.S and x is given by $= [x_B, 0]$, where $x_B = B^{-1}b$ and 0 is the $(n - m)$ component null vector.

We have to show that x is an extreme point of the convex set X of feasible solutions of the equation $Ax = b, x \geq 0$.

Let x be not an extreme point of the convex set X. Then there exist two points $x^/, x^{//},\ x^/ \neq x^{//}$ in X such that it is possible to express x as

$x = \lambda x^/ + (1 - \lambda)x^{//},\ 0 < \lambda < 1,$ where $x^/, x^{//}$ are given by

$x^/ = [u_1, v_1],\ x^{//} = [u_2, v_2]$ where u_1 contains m components of $x^/$, corresponding to the variables $x_1, x_2, \dots, x_m$ and v_1 contains the remaining $(n - m)$ components of $x^/$. Similarly u_2 and v_2 contains the first m and the remaining $(n - m)$ components of $x^{//}$ respectively.

Thus, $= \lambda[u_1, v_1] + (1 - \lambda)[u_2, v_2] = [\,\lambda u_1 + (1 - \lambda)u_2\,,\ \lambda v_1 + (1 - \lambda)v_2]$.

As $x = [x_B, 0]$, then equating the components corresponding to the last $(n - m)$ variables, we get $\lambda v_1 + (1 - \lambda)v_2 = 0$ which is possible only when $v_1 = 0$ and $v_2 = 0$ [as $v_1 \geq 0$, $v_2 \geq 0$ and $0 < \lambda < 1$].

Thus $x^/ = [u_1, 0],\ x^{//} = [u_2, 0]$. Hence u_1 and u_2 are the m components of the solution set corresponding to the basic variables $x_1, x_2, \dots, x_m$ for which the basis matrix is B. Then $u_1 = B^{-1}b$ and $u_2 = B^{-1}b$. Hence, $x_B = u_1 = u_2$. So the three points $x, x^/, x^{//}$ are not different and therefore x cannot be expressed as a convex combination of two distinct points. So a B.F.S is an extreme point.

Conversely, let us assume that x is an extreme point of the convex set X of feasible solutions of the equation $Ax = b, x \geq 0$.

We have to show that x is a B.F.S.

Let $x = [x_1, x_2, \dots, x_k, 0, \dots, 0]$, number of zero components are $n - k$, $x_j \geq 0$ for $j = 1,2, \dots, k$.

If the column vectors $a_1, a_2, \dots \dots, a_k$ associated with the variables $x_1, x_2, \dots, x_k$ respectively are L.I (which is possible only for $k \leq m$) then x, the extreme point of the convex set, is a B.F.S and we have nothing to prove.

If $a_1, a_2, \dots \dots, a_k$ are not L.I then $\sum_{j=1}^{k} a_j x_j = b$ and $\sum_{j=1}^{k} a_j \lambda_j = 0$ with at least one $\lambda_j \neq 0$.

Let $\delta > 0$, then from the above two equations we get $\sum_{j=1}^{k} (x_j \pm \delta \lambda_j) a_j = b$.

Consider δ such that $0 < \delta < l$, where $l = \min_j \left(\frac{x_j}{|\lambda_j|} \right)$, $\lambda_j \neq 0$.

Then $x_j \pm \delta \lambda_j \geq 0$ for $j = 1,2, \dots, k$.

Hence the two points

$x^{/} = [x_1 + \delta\lambda_1 , x_2 + \delta\lambda_2, \dots , x_k + \delta\lambda_k , 0, \dots, 0]$ and

$x^{//} = [x_1 - \delta\lambda_1, x_2 - \delta\lambda_2, \dots, x_k - \delta\lambda_k, 0, \dots, 0]$

are points of the convex set X.

Now, $\frac{1}{2}x^{/} + \frac{1}{2}x^{//} = x$, so x can be expressed as $x = \lambda x^{/} + (1 - \lambda)x^{//}$ where $\lambda = 1/2$.

Thus x is being expressed as a convex combination of two distinct points of X which contradicts the assumption that x is an extreme point. So the column vectors $a_1, a_2, \dots \dots, a_k$ are L.I, and hence x is a B.F.S.

Note: There is a one to one correspondence between the extreme points and B.F.S in case of non-degenerate B.F.S.

Eamples

1. In E^2, prove that the set $X = \{(x, y) | x + 2y \leq 5\}$ is a convex set.
   Solution: The set is non empty. Let (x_1, y_1) and (x_2, y_2) be two points of the set. Then $x_1 + 2y_1 \leq 5$ and $x_2 + 2y_2 \leq 5$.
   The convex combination of the two points is a point given by
   $[\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2],$.
   Now $\quad \lambda x_1 + (1-\lambda)x_2 + 2[\lambda y_1 + (1-\lambda)y_2]$
   $= \lambda(x_1 + 2y_1) + (1-\lambda)(x_2 + 2y_2) \leq 5\lambda + 5(1-\lambda) = 5$
   So the convex combination of the two points is $0 \leq \lambda \leq 1$ a point of the set. Thus the set is a convex set.
2. Prove that the set defined by $X = \{x : |x| \leq 2\}$ is a convex set.
   Solution: The set is non empty. Let x_1 and x_2 be two points of the set . Then $|x_1| \leq 2$ and $|x_2| \leq 2$.
   The convex combination of the two points is a point
   $x^* = \lambda x_1 + (1-\lambda)x_2, 0 \leq \lambda \leq 1$.
   Now $|\lambda x_1 + (1-\lambda)x_2| \leq |\lambda x_1| + |(1-\lambda)x_2|$
   $\leq |\lambda||x_1| + |(1-\lambda)||x_2| \leq 2\lambda + 2(1-\lambda) \leq 2$.
   Hence $x^* \in X$. So the set is a convex set.
3. Prove that in E^2, the set $X = \{(x, y) | x^2 + y^2 \leq 4\}$ is a convex set.
   Solution: Let (x_1, y_1) and (x_2, y_2) be two points of the set X.
   Then $x_1^2 + y_1^2 \leq 4$, and $x_2^2 + y_2^2 \leq 4$.

   The convex combination of the two points is a point given by

   $[\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2],$.
   Now $\quad [\lambda + (1-\lambda)x_2]^2 + [\lambda y_1 + (1-\lambda)y_2]^2$
   $= \lambda^2(x_1^2 + y_1^2) + (1 - x_1\lambda)^2(x_2^2 + y_2^2) + 2\lambda(1-\lambda)(x_1x_2 + y_1y_2)$
   $\leq \lambda^2(x_1^2 + y_1^2) + (1 - x_1\lambda)^2(x_2^2 + y_2^2)$
   $+ \frac{2\lambda(1-\lambda)(x_1^2 + y_1^2 + x_2^2 + y_2^2)}{2}$
   $\leq 4\lambda^2 + 4(1-\lambda)^2 + 8\lambda(1-\lambda)$, since $x_1^2 + y_1^2 + x_2^2 + y_2^2 \leq 8$

   $= 4$

   Therefore the point $\in X$. Hence the set is a convex set.

CHAPTER IV

Fundamental Properties of Simplex Method

Reduction of a F.S. to a B.F.S

Theorem: if a linear programming problem $Ax = b, x \geq 0,$ where A is the mxn coefficient matrix $(n > m),\ r(A) = m$ has one feasible solution, then it has at least one basic feasible solution.

Proof: Let $x = (x_1, x_2, \dots \dots, x_n)^T$ be a feasible solution to the set of equations $= b, x \geq 0$. Out of n components of the feasible solution, let k components be positive and the remaining $n - k$ components be zero $(1 \leq k \leq n)$ and we also make an assumption that the first k components are positive and the last $n - k$ components are zero.

Then $x = (x_1, x_2, \dots \dots, x_k, 0, \dots, 0)^T$, number of zeroes being $n - k.$

If $a_1, a_2, \dots \dots, a_k$ be the column vectors corresponding to the variables $x_1, x_2, \dots \dots, x_k$, then

$x_1 a_1 + x_2 a_2 + \cdots \dots + x_k a_k = b\ or, \sum_{j=1}^{k} x_j a_j = b$ …………… (1)

We will consider three cases

(i)$k \leq m$ and the column vectors $a_1, a_2, \dots \dots, a_k$ are linearly independent (L.I)

(ii) $k > m$

(iii) $k \leq m$ and the column vectors $a_1, a_2, \dots \dots, a_k$ are linearly dependent (L.D)

Case(i)If $k \leq m$ and the column vectors $a_1, a_2, \dots \dots, a_k$ are L.I , then by definition the F.S. is a B.F.S. If $= m$, the solution is a non degenerate B.F.S and if $k < m$, the solution is a degenerate B.F.S.

Case(ii)If $k > m$ and the columns $a_1, a_2, \dots \dots, a_k$ are L.D, the solution is not basic. By applying a technique given below, the number of positive components in the solution can be reduced one by one till the corresponding column vectors are L.I. (This will be possible as a set of one non-null vector is L.I.)

Procedure: As the column vectors $a_1, a_2, \dots \dots, a_k$ are L.D, there exist scalars $\lambda_j, j = 1,2, \dots, k$, not all zero, such that

$\lambda_1 a_1 + \lambda_2 a_2 + \cdots \dots + \lambda_k a_k = 0 \; or, \; \sum_{j=1}^{k} \lambda_j a_j = 0$ ……………….. (2)

Now at least one λ_j is positive (if not, multiply both sides of equation (1) by -1).

Let $=$ max$_j \left(\frac{\lambda_j}{x_j}\right), j = 1,2, \dots, k$,

As all $x_j > 0$ and $max\lambda_j > 0$, then v is essentially a positive quantity. Multiplying equation (2) by $1/v$ and subtracting from equation (1) we get

$\sum_{j=1}^{k} (x_j - \frac{\lambda_j}{v}) a_j = b$ ………………………….. (3)

which indicates that

$$x^{/} = \left[x_1 - \frac{\lambda_1}{v}, x_2 - \frac{\lambda_2}{v}, \dots \dots, x_k - \frac{\lambda_k}{v}, 0, \dots \dots, 0\right]$$

is a solution set of the equations $Ax = b$.

Now $\geq \frac{\lambda_j}{x_j}$. That implies $x_j \geq \frac{\lambda_j}{v}$ or, $x_j - \frac{\lambda_j}{v} \geq 0$, $x_j - \frac{\lambda_j}{v} = 0$ for at least one j.

Then $x_j^{/} = x_j - \frac{\lambda_j}{v} \geq 0, \; j = 1,2, \dots \dots, k$, at least one of a them is equal to zero.

Therefore $x^{/} = (x_1^{/}, x_2^{/}, \dots \dots, x_k^{/}, 0, \dots \dots, 0)$ is also a feasible solution of $Ax = b$ with maximum number of positive variables $k - 1$. By applying this method repeatedly we ultimately get a basic feasible solution.

(iii)In this case, as the vectors are L.D, we use the above procedure to get a B.F.S.

We state another theorem without proof.

Theorem (statement only) The necessary and sufficient condition that all basic solutions will exist and will be non-degenerate is that , every set of m column vectors of the augmented matrix $[A \; b]$ is linearly independent.

Problems

1. $x_1 = 1, x_2 = 3, x_3 = 2$ is a feasible solution of the equations
   $2x_1 + 4x_2 - 2x_3 = 10, \quad 10x_1 + 3x_2 + 7x_3 = 33$
   Reduce the above F.S to a B.F.S.
   Solution: The given equations can be written as $Ax = b$ where
   $A = [a_1 \;\; a_2 \;\; a_3] = \begin{bmatrix} 2 & 4 & -2 \\ 10 & 3 & 7 \end{bmatrix}$ and $(A) = 2$. Hence the two equations are L.I. , but a_1 , a_2 , a_3 are L.D. Hence there exist three constants $\lambda_1, \lambda_2, \lambda_3$, (not all zero) such that $\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = 0$,
   or, $\quad \lambda_1 \begin{bmatrix} 2 \\ 10 \end{bmatrix} + \lambda_2 \begin{bmatrix} 4 \\ 3 \end{bmatrix} + \lambda_3 \begin{bmatrix} -2 \\ 7 \end{bmatrix} = 0$ which gives

$2\lambda_1 + 4\lambda_2 - 2\lambda_3 = 0$ and $10\lambda_1 + 3\lambda_2 + 7\lambda_3 = 0$

By cross multiplication,

$\frac{\lambda_1}{34} = \frac{\lambda_2}{-34} = \frac{\lambda_1}{-34} = k = \frac{1}{34}$ (say).

Then we get $\lambda_1 = -1, \; \lambda_2 = 1, \; \lambda_3 = 1$.

Hence $-a_1 + a_2 + a_3 = 0$.

Therefore $v = \max_j \left(\frac{\lambda_j}{x_j}, \lambda_j > 0 \right) = max\left(\frac{1}{3}, \; \frac{1}{2} \right) = \frac{1}{2}$.

Hence a feasible solution is given by

$x^{/} = \left[x_1 - \frac{\lambda_1}{v}, x_2 - \frac{\lambda_2}{v}, x_3 - \frac{\lambda_3}{v} \right] = [1 + 2, 3 - 2, 2 - 2] = [3,1,0]$ which is a B.F.S.

2. Given (1,1,2) is a feasible solution of the equations
   $x_1 + 2x_2 + 3x_3 = 9, \quad 2x_1 - x_2 + x_3 = 3$
   Reduce the above F.S to one or more B.F.S.

Solution: The given equations can be written as $Ax = b$ where

$A = [a_1 \;\; a_2 \;\; a_3] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix}$ and $R(A) = 2$. The equations $Ax = b$ can be written as $x_1 a_1 + x_2 a_2 + x_3 a_3 = b$. As (1,1,2) is a solution of $Ax = b$, we have

$a_1 + a_2 + 2a_3 = b.$ ………… (1)
Hence the two equations are L.I. , but $a_1\,, a_2\,, a_3$ are L.D. So there exist three constants $\lambda_1, \lambda_2, \lambda_3$, (not all zero) such that $\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = 0$,

or, $\lambda_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 0$ which gives
$\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0$ and $2\lambda_1 - \lambda_2 + \lambda_3 = 0$

By cross multiplication,

$\frac{\lambda_1}{5} = \frac{\lambda_2}{5} = \frac{\lambda_1}{-5} = k = \frac{1}{5}$ (say).

Then we get $\lambda_1 = 1,\ \lambda_2 = 1,\ \lambda_3 = -1$.

Hence $a_1 + a_2 - a_3 = 0.$ ……………… (2)

Therefore $v = \max_j \left(\frac{\lambda_j}{x_j}\,, \lambda_j > 0 \right) = max \left(\frac{1}{1}\,,\ \frac{1}{1} \right) = 1$ which occurs at $j = 1{,}2$.

Thus we shall have to eliminate either a_1 or a_2 from the set of vectors $a_1\,, a_2\,, a_3$ to get a basis and hence a basic solution. Subtracting (2) from (1) we get,

$0a_1 + 0a_2 + 3a_3 = b$ which shows that (0,0,3) is a feasible solution and as $a_1\,, a_3$ and $a_2\,, a_3$ are L.I , the solution is a B.F.S.

Again taking $\frac{\lambda_1}{5} = \frac{\lambda_2}{5} = \frac{\lambda_1}{-5} = k^{/} = \frac{1}{-5}$, we get $-a_1 - a_2 + a_3 = 0$ which gives another B.F.S. as (3,3,0).

Fundamental Theorem of Linear Programming:

Statement: If a L.P.P. , optimize $z = c^T x$ subject to $Ax = b\,,\ x \geq 0$, where A is the $m\ x\ n$ coefficient matrix $(n > m)\,, r(A) = m$, has an optimal solution then there exists at least one B.F.S. for which the objective function will be optimal.

Proof: It is sufficient to consider a maximization problem as a minimization problem can be converted into a maximization problem.

Let $x = [x_1, x_2, \dots \dots, x_n]^{\,T}$ be an optimal feasible solution to the problem which makes the objective function maximum. Out of $x_1, x_2, \dots \dots, x_n$ let k components $(1 \leq k \leq n)$ are positive and the remaining $(n - k)$ components are zero. We further make an assumption that the first k components are positive. Thus the optimal solution is $x = [x_1, x_2, \dots, x_k, 0, \dots, 0]^T$, $(n-k)$ zero components, and $\overline{z} = \overline{c^T x} = \sum_{j=1}^{k} c_j x_j$.

If $a_1, a_2, \dots\dots, a_k$ be the column vectors associated with the variables $x_1, x_2, \dots, x_k$ then the optimal solution will be a B.F.S provided the vectors $a_1, a_2, \dots\dots, a_k$ are L.I. This is possible only if $k \leq m$.

We know $\sum_{j=1}^{k} x_j a_j = b, x_j \geq 0, j = 1,2, \dots, k.$ ………… (1)

Let us assume that $a_1, a_2, \dots\dots, a_k$are L.D.

Then $\sum_{j=1}^{k} \lambda_j a_j = 0$ with at least one $\lambda_j > 0$. …………. (2)

Taking $v = \max_j \left(\frac{\lambda_j}{x_j}\right)$ which is a positive quantity, and the solution set $\overline{x^/} = \left[x_1^/, x_2^/, \dots, x_k^/, 0, \dots, 0\right]^T$ where $x_j^/ = x_j - \frac{\lambda_j}{v} \geq 0,\ j = 1,2, \dots\dots, k$ which contains maximum $k-1$ positive components. Let $x_k^/ = 0$, then $\overline{x^/} = \left[x_1^/, x_2^/, \dots, x_{k-1}^/, 0, \dots, 0\right]^T$, $(n-k+1)$ zero components. The value of the objective function for this solution set is

$$\overline{z_1} = \sum_{j=1}^{k-1} c_j x_j^/ = \sum_{j=1}^{k} c_j x_j^/ \quad (\text{as } x_k^/ = 0)$$

$$= \sum_{j=1}^{k} c_j \left(x_j - \frac{\lambda_j}{v}\right) = \sum_{j=1}^{k} c_j x_j - \frac{1}{v} \sum_{j=1}^{k} c_j \lambda_j = \overline{z} - \frac{1}{v} \sum_{j=1}^{k} c_j \lambda_j .$$

If $\sum_{j=1}^{k} c_j \lambda_j$ =0 (which will be proved at the end) , then $\overline{z_1} = \overline{z}$ and the solution set $\overline{x^/}$ is also an optimal solution. If the column vectors corresponding to $x_1^/, x_2^/, \dots, x_{k-1}^/$ are L.I. then the optimal solution $\overline{x^/}$ is a B.F.S. If the column vectors are L.I then repeating the above procedure at most a finite number of times we will finally get a B.F.S (as a single non-null vector is L.I.) which is also an optimal solution.

To prove that $\sum_{j=1}^{k} c_j\lambda_j = 0$, if possible let $\sum_{j=1}^{k} c_j\lambda_j \neq 0$.
Then $\sum_{j=1}^{k} c_j\lambda_j > 0 \; or < 0$. We multiply $\sum_{j=1}^{k} c_j\lambda_j$ by a quantity $\delta(\neq 0)$, such that $\delta \sum_{j=1}^{k} c_j\lambda_j > 0$.

Hence $\sum_{j=1}^{k} c_j x_j + \delta \sum_{j=1}^{k} c_j\lambda_j > \sum_{j=1}^{k} c_j x_j$, or, $\sum_{j=1}^{k} c_j(x_j + \delta\lambda_j) > \bar{z}$. …. (3)

Multiplying equation (2) by δ and adding to (1) we get $\sum_{j=1}^{k}(x_j + \delta\lambda_j)a_j = b$, which shows that $(x_j + \delta\lambda_j), j = 1,2,\dots,k$ is a solution set of the system $Ax = b$. Value of δ is given by the relation

$$\max_j\left(-\frac{x_j}{\lambda_j}\ ,\ \lambda_j > 0\right) \leq \delta \leq \min_j\left(-\frac{x_j}{\lambda_j}\ ,\ \lambda_j < 0\right).$$

As for $\lambda_j > 0, x_j + \delta\lambda_j > 0$ gives $\delta \geq -\frac{x_j}{\lambda_j}$ and for $\lambda_j < 0, x_j + \delta\lambda_j > 0$ gives $\leq -\frac{x_j}{\lambda_j}$.

Hence for particular values of δ it is always possible to get $x_j + \delta\lambda_j \geq 0$ for all j. So the solution set $(x_j + \delta\lambda_j), j = 1,2,\dots,k$ is a feasible solution of the system $Ax = b$. From (3) it is clear that this solution set gives the value of the objective function greater than $\bar{z}$ which contradicts the fact that $\bar{z}$ is the maximum value of the objective function.

Hence $\sum_{j=1}^{k} c_j\lambda_j = 0$.

CHAPTER V

Simplex Method

After introduction of the slack and surplus variables and by proper adjustment of z, let us consider the L.P.P. as

Maximize $z = c^T x$ subject to $Ax = b, x \geq 0$,

where A is the mxn coefficient matrix given by

$A = (a_1, a_2, \dots \dots, a_n)$,

where $a_j = (a_{1j}, a_{2j}, \dots \dots, a_{mj})^T$ is a column vector associated with the vector x_j , $j = 1,2, \dots \dots, n$.

$c = (c_1, c_2, \dots \dots, c_r, 0,0, \dots ,0)^T$ is a n-component column vector,

$x = (x_1, x_2, \dots \dots, x_r, x_{r+1} , x_{r+2} , \dots \dots, x_n)^T$ is a n-component column vector, where $x_{r+1} , x_{r+2} , \dots \dots, x_n$ are either slack or surplus variables and

$b = (b_1, b_2, \dots \dots, b_m)^T$ is a m-component column vector.

We make two assumptions: components of b are non negative by proper adjustments and $m < n$ (this assumption is non restrictive).

As none of the m converted equations are redundant then there exists at least one set of m column vectors, say, $\beta_1, \beta_2, \dots, \beta_m$ of the coefficient matrix A which are linearly independent. Then one basis matrix B which is a submatrix of A is given by $B = [\beta_1 \ \beta_2 \dots \beta_m]$.

Let $x_{B1}, x_{B2}, \dots, x_{Bm}$ be the variables associated with the basic vectors $\beta_1, \beta_2, \dots, \beta_m$ respectively. Then the basic variable vector is $x_B = [x_{B1} \ x_{B2} \ \dots x_{Bm}]^T$.

The solution set corresponding to the basic variables is $x_B = B^{-1} b$.

We assume that $x_B \geq 0$, i.e. the solution is a B.F.S.

Let $c_{B1}, c_{B2}, \dots, c_{Bm}$ be the coefficients of $x_{B1}, x_{B2}, \dots, x_{Bm}$ respectively in the objective function $= c^T x$, then $c_B = [c_{B1}\ c_{B2}\ \dots\ c_{Bm}]^T$ is an m component column vector known as the associated cost vector.

Now a value z_B is defined as $z_B = c_{B1}\, x_{B1} + c_{B2} x_{B2} + \cdots + c_{Bm} x_{Bm} = c_B^T x_B$.

z_B is the value of the objective function corresponding to the B.F.S, where the basis matrix is B .

Now, $\beta_1, \beta_2, \dots, \beta_m$ are L.I and so is a basis of E^m. Therefore all the vectors a_j can be expressed as a linear combination of $\beta_1, \beta_2, \dots, \beta_m$.

Let $a_j = \beta_1 y_{1j} + \beta_2 y_{2j} + \cdots + \beta_m y_{mj} = B y_j$ where $y_j = \left[y_{1j}\ \ y_{2j}\ \ \dots\ \ y_{mj}\right]^T$, Therefore $y_j = B^{-1} a_j$.

Net evaluation: Evaluation is defined as ${c_B}^T y_j$ which is usually denoted by z_j . So z_j is given by $z_j = {c_B}^T y_j = {c_B}^T B^{-1} a_j = c_{B1} y_{1j} + c_{B2} y_{2j} + \cdots + c_{Bm} y_{mj}$ and $z_j - c_j$ is called the net evaluation.

If the coefficient matrix A contains m unit column vectors which are L.I, then this set of vectors constitute a basis matrix. Let $e_1, e_2, \dots, e_i, \dots, e_m$ be m independent unit vectors of the coefficient matrix, all of which may not be placed in the ascending order of i $(i = 1,2, \dots, m)$. For example, e_1, e_2, e_3 may occur at the 5th ,7th , 3rd column of A respectively. But the basis matrix B is the identity matrix. Hence the components of the solution set corresponding to the basic variables are $x_{Bi} = b_i$, $i = 1,2, \dots, m$ and $y_j = B^{-1} a_j = a_j$, that is the vectors y_j are nothing but the column vectors a_j due to this transformation.

Note: In the simplex method all equations are adjusted so that the basis matrix is the identity matrix and $b_i \geq 0$ for all i.

Optimality test: For a maximization problem, if at any stage, $z_j - c_j \geq 0$ for all j then the current solution is optimal. If $z_j - c_j < 0$ for at least one j and for this j at least one $y_{ij} > 0$, then the value of the objective function can be improved further. If any $z_j - c_j < 0$ and $y_{ij} \leq 0$ for all i then the problem has no finite optimal value and the problem is said to have an unbounded solution.

Selection of a vector to enter the next basis and a vector to leave the previous basis : If $z_j - c_j < 0$ for at least one j and for this j at least one $y_{ij} > 0$, then we shall have to select a new basis. Thus one new vector is to be selected from a_j (which is not in the previous basis) to replace a vector in the previous basis to form a new basis.

If $z_k - c_k = \min_j\{z_j - c_j \ , z_j - c_j < 0\}$, then a_k is the vector to enter in the new basis and the k^th^ column of the simplex table is called the key column or the pivot column. If the minimum occurs for more than one value of j then the selection is arbitrary.

Let a_k be the vector to enter in the new basis.

If $\min_i\{\frac{x_{Bi}}{y_{ik}}, y_{ik} > 0\} = y_{rk}$, then the vector in the r^th^ position of the current basis will be replaced by a_k . The r^th^ row of the table is called the key row and y_{rk} is called the key element. If the value of r is not unique, then again the choice is arbitrary.

We will illustrate the simplex method in details through examples.Before going into the examples, we state and prove one more theorem.

Theorem: Minimum value of z is the negative of the maximum value of $(-z)$ with the same solution set. In other words, $min\ z = -max(-z)$ with the same solution set.

Proof: Let $z = c^T x$ attain its minimum at $x = x_0$ then $z = c^T x_0$.

Hence $c^T x \geq c^T x_0$ or, $-c^T x \leq -c^T x_0$.

Therefore,

$\max(-c^T x) = -c^T x_0$ or, $c^T x_0 = -\max(-c^T x)$ or, $min\ z = -max(-z)$,

with the same solution set. Similarly, $max\ z = -min\ (-z)$.

Examples

1. Solve the L.P.P.
   Maximize $z = 5x_1 + 2x_2 + 2x_3$

Subject to $x_1 + 2x_2 - 2x_3 \leq 30$

$x_1 + 3x_2 + \ \ x_3 \leq 36\ \ , x_1, x_2, x_3 \geq 0$.

Solution: This is a maximization problem , $b_i \geq 0, i = 1{,}2$ and the constraints are both " $\leq$ " type. So introducing two slack variables $x_4, x_5,$ one to each constraint, we get the following converted equations.

$$\begin{aligned} &x_1 + 2x_2 - 2x_3 + x_4 \qquad = 30 \\ &x_1 + 3x_2 + \ \ x_3 \qquad + x_5 = 36\ \ , x_1, x_2, x_3, x_4, x_5 \geq 0\ . \end{aligned}$$

The adjusted objective function is

$z = 5x_1 + 2x_2 + 2x_3 + 0x_4 + 0x_5$. In notations, the new problem is

Max $z = c^T x$ subject to $Ax = b, x \geq 0,$ where $A = [a_1\ a_2\ \ a_3\ a_4\ a_5]$,

$a_1 = \begin{bmatrix}1\\1\end{bmatrix}, a_2 = \begin{bmatrix}2\\3\end{bmatrix}, a_3 = \begin{bmatrix}-2\\1\end{bmatrix}, a_4 = \begin{bmatrix}1\\0\end{bmatrix}, a_5 = \begin{bmatrix}0\\1\end{bmatrix}, b = \begin{bmatrix}30\\36\end{bmatrix},$

$x_B = \begin{bmatrix}x_{B1}\\x_{B2}\end{bmatrix} = \begin{bmatrix}x_4\\x_5\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix} = \begin{bmatrix}30\\36\end{bmatrix}, c_B = \begin{bmatrix}c_{B1}\\c_{B2}\end{bmatrix} = \begin{bmatrix}c_4\\c_5\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = 0,$

$z_B = c_B^T x_B = c_{B1}\, x_{B1} + c_{B2} x_{B2} = 0, y_j = B^{-1} a_j = I_2 a_j = a_j$ that is $y_{ij} = a_{ij}$. With the above information we now proceed to construct the initial simplex table.

Initial simplex table

		c	5	2	2	0	0	
Basis	c_B	b	a_1	a_2	a_3	$a_4(e_1)$	$a_5(e_2)$	Min ratio= $\frac{x_{Bi}}{y_{i1}}, y_{i1} > 0$
$a_4{}^*$	0	30	1^*	2	-2	1	0	$\frac{30}{1} = 30^*$ →
a_5	0	36	1	3	1	0	1	$\frac{36}{1} = 36$
$z_j - c_j$		$z_0 = 0$	-5^*	-2	-2	0	0	
			↑					

Rule of construction of the second table: The new basis is a_1 and a_5 and therefore they must be the columns of the identity matrix. We make the necessary row operations as follows:

R_1 is the key row, a_1 is the key column, y_{11} is the key element.

$R_1^{/}$ = new key row = $\frac{1}{y_{11}} R_1$

$R_2^{/} = R_2 - y_{11} R_1^{/}$

The same notations will be used in all the tables but the entries will keep changing.

Second simplex table (1st iteration)

		C	5	2	2	0	0	
Basis	c_B	B	a_1	a_2	a_3	a_4	a_5	Min ratio = $\frac{x_{Bi}}{y_{i3}}, y_{i3} > 0$
a_1	5	30	1	2	-2	1	0	$- -$
$a_5{}^*$	0	6	0	1	3*	-1	1	$\frac{6}{3} = 2^*$ →
$z_j - c_j$			$z_0 = 0$	8	-12^*	5	0	

$z = 150$ ↑

R_2 is the key row, a_3 is the key column, y_{23} is the key element.

$R_2^{/}$ = new key row = $\frac{1}{y_{23}} R_2$

$R_1^{/} = R_1 - y_{13} R_2^{/}$

Third simplex table (2nd iteration)

		C	5	2	2	0	0
Basis	c_B	B	a_1	a_2	a_3	a_4	a_5
a_1	5	34	1	8/3	0	1/3	2/3
a_3	2	2	0	1/3	1	-1/3	1/3
$z_j - c_j$			0	12	0	1	4

Z=174

Here $z_j - c_j \geq 0$ for all j. Hence the solution is optimal.

$z_{max} = 174$ for $x_1 = 34,\ x_2 = 0,\ x_3 = 2$.

2. Solve the L.P.P. by simplex method
   Maximize $z = 4x_1 + 7x_2$
   Subject to $\begin{array}{rl} 2x_1 + \ \ x_2 \leq & 1000 \\ x_1 + \ \ x_2 \leq & \ \ 600 \\ -x_1 - 2x_2 \geq & -1000 \, , \quad x_1, x_2 \geq 0 \, . \end{array}$

   This is a maximization problem. Multiplying the third constraint by (-1) we get $x_1 + 2x_2 \leq 1000$. Hence all $b_i \geq 0$ and all constraints are "≤" type. Introducing three slack variables x_3, x_4, x_5, one to each constraint we get the following converted equations

   $$\begin{array}{llll} 2x_1 + \ x_2 + \ x_3 & & = 1000 & \\ \ x_1 + \ x_2 + & +x_4 & = \ \ 600 & \\ \ x_1 + 2x_2 \ + & \qquad +x_5 & = 1000 & , x_1, x_2, x_3, x_4, x_5 \geq 0 \, . \end{array}$$

   The adjusted objective function is $= 4x_1 + 7x_2 + 0x_3 + 0x_4 + 0x_5$.

   In notations, the new problem is

   Max $z = c^T x$ subject to $Ax = b, x \geq 0$, where $A = [a_1\ a_2\ a_3\ a_4\ a_5]$,

   $$a_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, a_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, a_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, a_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, a_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, b = \begin{bmatrix} 1000 \\ 600 \\ 1000 \end{bmatrix},$$

   $$x_B = \begin{bmatrix} x_{B1} \\ x_{B2} \\ x_{B3} \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1000 \\ 600 \\ 1000 \end{bmatrix}, c_B = \begin{bmatrix} c_{B1} \\ c_{B2} \\ c_{B3} \end{bmatrix} = \begin{bmatrix} c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

   $z_B = c_B^T x_B = c_{B1}\, x_{B1} + c_{B2}x_{B2} + c_{B3}x_{B3} = 0, y_j = B^{-1}a_j = I_3 a_j = a_j$ that is $y_{ij} = a_{ij}$. With the above information we now proceed to construct the initial simplex table.

		C	4	7	0	0	0	
Basis	c_B	b	a_1	a_2	a_3	a_4	a_5	Min ratio= $\frac{x_{Bi}}{y_{i1}}, y_{i1} > 0$
a_3	0	1000	2	1	1	0	0	$\frac{1000}{1}$=1000
a_4	0	600	1	1	0	1	0	$\frac{600}{1} = 600$
$a_5{}^*$	0	1000	1	2^*	0	0	1	$\frac{1000}{2} = 500^*$
$z_j - c_j$		$z = 0$	-4	-7^*	0	0	0	
y_{32} is the key element. $R_3^/ =$ new key row $= \frac{1}{y_{32}} R_3$, $R_i^/ = R_i - y_{i2}R_3^/$, $i = 1,2.$								
a_3	0	500	3/2	0	1	0	-1/2	$\frac{500}{3/2}$=1000/3
$a_4{}^*$	0	100	$-\frac{1}{2}^*$	1	0	1	-1/2	$\frac{100}{1/2} = 200^*$
a_2	7	500	1/2	0	0	0	1/2	$\frac{500}{1/2}$ =1000
$z_j - c_j$		3500	$-\frac{1}{2}^*$	0	0	0	7/2	
y_{21} is the key element. $R_2^/ =$ new key row $= \frac{1}{y_{21}} R_2$, $R_i^/ = R_i - y_{i1}R_2^/$, $i = 1,3.$								
a_3	0	200	0	0	1	-3	1	
a_1	4	200	1	0	0	2	-1	
a_2	7	400	0	1	0	-1	1	
$z_j - c_j$		3600	0	0	0	1	3	

Here $z_j - c_j \geq 0$ for all j. Hence the solution is optimal.

$z_{max} = 3600$ for $x_1 = 200,\ x_2 = 400$.

3. Solve the L.P.P.
   Minimize $z = -2x_1 + 3x_2$
   Subject to $2x_1 - 5x_2 \leq 7$
   $4x_1 + x_2 \leq 8$
   $7x_1 + 2x_2 \leq 16,$ $x_1, x_2 \geq 0$.
   Solution: This is a minimization problem. Let $z^/ = -z$. Then $\min z =$

$-\max(-z) = -max\, z^/$. We solve the problem for $max\, z^/$ and the required $\min z = -\, max\, z^/$. Introducing two slack variables the converted equations are $2x_1 - 5x_2 + x_3 \quad = 7$
$4x_1 + \quad x_2 \quad + x_4 \quad = 8$
$7x_1 + 2x_2 \quad + x_5 = 16, \qquad x_1, x_2, x_3, x_4, x_5 \geq 0$.
The adjusted objective function is $z^/ = -2x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5$. The initial basis $B = (a_3, a_4, a_5) = I_3$ and we start the simplex table and solve the problem. We solve the problem in a compact manner as shown below.

		c	-2	3	0	0	0	
Basis	c_B	b	a_1	a_2	a_3	a_4	a_5	Min ratio$=\frac{x_{Bi}}{y_{i1}}, y_{i1} > 0$
a_3	0	7	2	-5	1	0	0	$\frac{7}{2}$
$a_4{}^*$	0	8	4*	1	0	1	0	$\frac{7}{2}$
a_5	0	16	7	2	0	0	1	$\frac{16}{7}$
$z_j - c_j$		$z^/ = 0$	-2^*	3	0	0	0	
y_{21} is the key element. $R_2^/ =$ new key row $= \frac{1}{y_{21}} R_2$, $R_i^/ = R_i - y_{i2} R_2^/$, $i = 1,2.$								
a_3	0	3	0	-11/2	1	-1/2	0	
a_1	2	2	1	¼	0	¼	0	
a_5	0	2	0	¼	0	-7/4	1	
$z_j - c_j$		$z^/$ =4	0	7/2	0	½	0	

Here $z_j - c_j \geq 0$ for all j. Therefore the solution is optimal. Hence $z^/ = 4$. Now $\min z = -\, max\, z^/ = -4$. Hence the minimum value of z is -4 corresponding to the optimal basic feasible solution. $x_B = [x_3\ x_1\ x_5] = [3\ 2\ 2]$, i.e., for $x_1 = 2, x_2 = 0$, the objective function of the original problem attains its minimum value. This solution is a degenerate B.F.S.

4. Use simplex method to solve the L.P.P

Maximize $z = 2x_2 + x_3$

Subject to $\quad x_1 + x_2 - 2x_3 \leq 7$

$-3x_1 + x_2 + 2x_3 \leq 3 \quad , \quad x_1, x_2, x_3 \geq 0$.

Solution: Adding two slack variables x_4, x_5, one to each constraint, the converted equations are

$$x_1 + x_2 - 2x_3 + x_4 \quad = 7$$

$$-3x_1 + x_2 + 2x_3 \quad + x_5 = 3, \qquad x_j \geq 0 \quad \text{for} \quad j = 1, \ldots, 5.$$

The adjusted objective function is $z = 0x_1 + 2x_2 + x_3 + 0x_4 + 0x_5$.

$b = \begin{bmatrix} 7 \\ 3 \end{bmatrix} \geq 0, B = \begin{bmatrix} a_4 \\ a_5 \end{bmatrix} = I_2$ is the initial unit basis matrix.

		C	0	2	1	0	0	
Basis	c_B	B	a_1	a_2	a_3	$a_4(e_1)$	$a_5(e_2)$	Min ratio= $\frac{x_{Bi}}{y_{i1}}, y_{i1} > 0$
a_4	0	7	1	1	-2	1	0	$\frac{7}{1} = 7$
$a_5{}^*$	0	3	-3	1*	2	0	1	$\frac{3}{1} = 3^*$
$z_j - c_j$		$z = 0$	0	-2^*	-1	0	0	
y_{22} is the key element. $R_2^/$ = new key row = $\frac{1}{y_{22}} R_2$, $R_1^/ = R_1 - y_{12} R_2^/$.								
$a_4{}^*$	0	4	4*	0	-4	1	-1	$\frac{4}{4} = 1^*$
a_2	2	3	-3	1	2	0	1	---
$z_j - c_j$		$z = 6$	-6^*	0	3	0	2	
y_{11} is the key element. $R_1^/$ = new key row = $\frac{1}{y_{11}} R_1$, $R_2^/ = R_2 - y_{21} R_1^/$.								
a_1	0	1	1	0	-1	¼	-1/4	
a_2	2	6	0	1	-1	¾	¼	
$z_j - c_j$		$z = 12$	0	0	-3	3/2	½	

In the third table $z_j - c_j < 0$ for $= 3$. But $y_{i3} < 0$ for all . Hence the problem has unbounded solution.

5. Use simplex method to solve the L.P.P

   Maximize $z = 5x_1 + 2x_2$

Subject to $6\,x_1 + 10x_2 \leq 30$

$$10x_1 + 4x_2 \leq 20 \ , x_1, x_2 \geq 0 \ .$$

Show that the solution is not unique. Write down a general form of all the optimal solutions.

Solution: Adding two slack variables x_3,x_4, one to each constraint, the converted equations are $6\,x_1 + 10x_2 + x_3 \qquad = 30$

$$10x_1 + 4x_2 \qquad + x_4 = 20, \qquad x_j \geq 0 \quad \text{for} \quad j = 1,2,3,4.$$

The adjusted objective function is $z = 5x_1 + 2x_2 + 0x_3 + 0x_4$.

$b = \begin{bmatrix} 30 \\ 20 \end{bmatrix} \geq 0, B = \begin{bmatrix} a_3 \\ a_4 \end{bmatrix} = I_2$ is the initial unit basis matrix.

Simplex tables		C	5	2	0	0	
Basis	c_B	B	a_1	a_2	a_3	$a_4(e_1)$	Min ratio= $\frac{x_{Bi}}{y_{i1}}, y_{i1} > 0$
a_3	0	30	6	10	1	0	$\frac{30}{6} = 5$
$a_4{}^*$	0	20	10^*	4	0	1	$\frac{20}{10} = 2^*$
$z_j - c_j$		$z = 0$	-5^*	-2	0	0	
y_{21} is the key element. $R_2^{/}$ = new key row $= \frac{1}{y_{21}} R_2$, $R_1^{/} = R_1 - y_{12} R_2^{/}$							
$a_3{}^*$	0	18	0	$\left(\frac{38}{5}\right)^*$	1	-3/5	$\frac{18}{38/5} = \frac{45}{19}^*$
a_1	5	2	1	2/5	0	1/10	$\frac{2}{2/5} = 5$
$z_j - c_j$		$z = 10$	0	0^*	0	½	
y_{12} is the key element. $R_1^{/}$ = new key row $= \frac{1}{y_{12}} R_1$, $R_2^{/} = R_2 - y_{22} R_1^{/}$							
a_2	2	45/19	0	1	5/38	-3/38	
a_1	5	20/19	1	0	-2/19	5/38	
$z_j - c_j$		$z = 10$	0	0	0	½	

In the second table, $z_j - c_j \geq 0$ for all j. Therefore the solution is optimal and $max\ z = 10$ at $x_1 = 2, x_2 = 0$. But $z_2 - c_2 = 0$ corresponding to a non-basic vector a_2 . Thus the solution is not unique. Using a_2 to enter in the next basis, the third table gives the same value of z but for $x_1 = 20/19, x_2 = 45/19$. We know that if there exists more than one optimal solution, then there exist an infinite number of optimal solutions, given bythe convex combination of the optimal solutions $x^/ = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $x^{//} = \begin{bmatrix} 20/19 \\ 45/19 \end{bmatrix}$. Hence all the optimal solutions are given by $\lambda x^/ + (1-\lambda)x^{//}, 0 \leq \lambda \leq 1,$

$$= \lambda \begin{bmatrix} 2 \\ 0 \end{bmatrix} + (1-\lambda) \begin{bmatrix} 20/19 \\ 45/19 \end{bmatrix}$$

These solutions are called alternative optima.

Artificial variables

To solve a problem by simplex method, we rewrite all the constraints as equations by introducing slack/surplus variables. We consider the following example.

Maximize $z = 2x_1 + 3x_2 - 4x_3$

Subject to $4x_1 + 2x_2 - x_3 \leq 4$

$$-3x_1 + 2x_2 + 3x_3 \geq 6$$

$$x_1 + x_2 - 3x_3 = 8 , x_j \geq 0, j = 1,2,3 .$$

First constraint is ≤ type and the second one is a ≥ type, so adding a slack and a surplus variable respectively, the two constraints are converted into equations. Hence the transformed problem can be written as

$$\begin{aligned} &\text{Maximize } z = 2x_1 + 3x_2 - 4x_3 + 0x_4 + 0x_5 \\ &\text{Subject to } 4x_1 + 2x_2 - x_3 + x_4 = 4 \\ &-3x_1 + 2x_2 + 3x_3 - x_5 = 6 \\ &x_1 + x_2 - 3x_3 = 8 , x_j \geq 0, j = 1,2,3,4,5 . \end{aligned}$$

To get the initial B.F.S for using the simplex method, we require an identity matrix as a sub-matrix of the coefficient matrix. To get that, we need to introduce some more variables which will be called the artificial variables. Even if a constraint is given as an equation, we still add an artificial variable (A.V) to get an initial B.F.S. So after introducing artificial variables the above problem is written as$4x_1 +$

$$2x_2 - x_3 + x_4 \qquad = 4$$

$$-3x_1 + 2x_2 + 3x_3 \qquad - x_5 + x_6 \qquad = 6$$

$$x_1 + x_2 - 3x_3 \qquad + x_7 = 8 \text{ , } x_j \geq 0, j = 1, \dots ,7 \text{ .}$$

Then the basis matrix is $[a_4 \;\; a_6 \;\; a_7] = I_2$.

In an attempt to solve a problem involving artificial variables by using simplex method, the following three cases may arise.

1. No artificial variables are present in the basis at the optimal stage indicates that all A.V.s are at the zero level and hence the solution obtained is optimal.
2. At the optimal stage, some artificial variables are present in the basis at the positive level indicates that there does not exist a F.S to the problem.
3. At the optimal stage, some artificial variables are present in the basis but at zero level indicates that some constraints are redundant.

Problems involving artificial variables can be solved by Charnes method of penalties or Big M-Method.

Charnes method of penalties or Big M-Method

In this method, after rewriting the constraints by introducing slack, surplus and artificial variables, we adjust the objective function by assigning a large negative cost, say $-M$ to each artificial variable. In the example given above, the objective function becomes Maximize $z = 2x_1 + 3x_2 - 4x_3 + 0x_4 + 0x_5 - Mx_6 - Mx_7$.

We then solve the problem using simplex method as explained earlier, the only point to remember is that once an artificial variable leaves a basis we drop the column corresponding to the vector associated with that A.V.

6. Solve the L.P.P
   Maximize $z = 2x_1 - 3x_2$
   Subject to $\quad -x_1 + \;\; x_2 \geq -2$
   $\qquad 5x_1 + 4x_2 \leq \; 46$

$7x_1 + 2x_2 \ \geq 32$, $x_j \geq 0, j = 1,2$.

Solution: In the first constraint $b_1 = -2 < 0$, so making it positive, $x_1 - x_2 \leq 2$. Introducing slack and surplus variables the converted equations are

$x_1 - \ x_2 + x_3 \qquad = \ 2$

$5x_1 + 4x_2 \quad + x_4 \quad = 46$

$7x_1 + 2x_2 \qquad - x_5 = 32$, $x_j \geq 0, j = 1, \dots ,5$ .The coefficient matrix does not contain a unit basis matrix, so we introduce an A.V in the third constraint and the set of equations are

$x_1 - \ x_2 + x_3 \qquad = \ 2$

$5x_1 + 4x_2 \quad + x_4 \qquad = 46$

$7x_1 + 2x_2 \qquad - x_5 + x_6 = 32$, $x_j \geq 0, j = 1, \dots ,6$

The adjusted objective function is $z = 2x_1 - 3x_2 + 0x_3 + 0x_4 + 0x_5 - Mx_6$ assigning very large negative price to the A.V.

		c	2	-3	0	0	0	-M	
Basis	c_B	b	a_1	a_2	a_3	a_4	a_5	a_6	Min ratio= $\frac{x_{Bi}}{y_{i1}}, y_{i1} > 0$
$a_3{}^*$	0	2	1^*	-1	1	0	0	0	$\frac{2}{1} = 2^*$
a_4	0	46	5	4	0	1	0	0	$\frac{46}{5} = 9\frac{1}{5}$
a_6	*-M*	32	7	2	0	0	-1	1	$\frac{32}{7} = 4\frac{4}{7}$
$z_j - c_j$			$-7M - 2^*$	-2M +3	0	0	M	0	
y_{11} is the key element. $R_1^{/}$ = new key row $= \frac{1}{y_{11}} R_1$, $R_i^{/} = R_i - y_{i1} R_1^{/}$,$i = 2,3$									

Table 2

		c	2	-3	0	0	0	-M	
Basis	c_B	b	a_1	a_2	a_3	a_4	a_5	a_6	Min ratio= $\frac{x_{Bi}}{y_{i1}}, y_{i1} > 0$
a_1	2	2	1	-1	1	0	0	0	$--$
a_4	0	36	0	9	-5	1	0	0	$\frac{36}{9} = 4$
$a_6{}^*$	*-M*	18	0	9^*	-7	0	-1	1	$\frac{18}{9} = 2^*$
$z_j - c_j$			0	$-9M + 1^*$	7M +2	0	M	0	
y_{32} is the key element. $R_3^{/}$ = new key row $= \frac{1}{y_{31}} R_3$, $R_i^{/} = R_i - y_{i2} R_3^{/}$, $i = 1,2$									

Table 3

		c	2	-3	0	0	0
Basis	c_B	b	a_1	a_2	a_3	a_4	a_5
a_1	2	4	1	0	2/9	0	-1/9
a_4	0	18	0	0	2	1	1
a_2	-3	2	0	1	-7/9	0	-1/9
$z_j - c_j$		z=2	0	0	25/9	0	1/9

As $z_j - c_j \geq 0$ for all j, optimality condition is reached. The artificial vector a_4 is not present in the final basis. Therefore the A.V x_6 is zero at the final stage. Hence the optimal solution obtained is a B.F.S and the maximum value of z is 2 for $x_1 = 4, x_2 = 2$.

7. Solve the L.P.P

$$\begin{aligned} &\text{Maximize } z = x_1 + 2x_2 \\ &\text{Subject to} \qquad x_1 - 5x_2 \leq 10 \\ &\qquad\qquad\qquad 2x_1 - \ \ x_2 \geq \ 2 \\ &\qquad\qquad\qquad\ x_1 + \ \ x_2 = 10\ , \ \ x_j \geq 0, j = 1,2\ . \end{aligned}$$

Soln: Introducing slack, surplus and artificial variables, the converted problem is

Maximize $z = 2x_1 - 3x_2 + 0x_3 + 0x_4 - Mx_5 - Mx_6$

subject to

$$\begin{aligned} x_1 - 5x_2 + x_3 \qquad\qquad\qquad &= \ 10 \\ 2x_1 - \ x_2 \qquad - x_4 + x_5 \qquad &= \ \ 2 \\ x_1 + \ x_2 \qquad\qquad\qquad + x_6 &= \ 10\ , \ \ x_j \geq 0, j = 1, \ldots, 6 \end{aligned}$$

x_3 is a slack variable, x_4 is a surplus variable and x_5, x_6 are artificial variables. We now construct the simplex tables and solve the problem.

		c	2	-3	0	0	-M	-M	
Basis	c_B	b	a_1	a_2	a_3	a_4	a_5	a_6	Min ratio= $\frac{x_{Bi}}{y_{i1}}, y_{i1} > 0$
a_3	0	10	1	-5	1	0	0	0	$\frac{10}{1}$ =10
$a_5{}^*$	-M	2	2^*	-1	0	-1	1	0	$\frac{2}{2} = 1^*$
a_6	-M	10	1	1	0	0	0	1	$\frac{10}{1}$ =10
$z_j - c_j$			$-3M - 1^*$	-2	0	M	0	0	
y_{21} is the key element. $R_2^/$ = new key row $= \frac{1}{y_{21}} R_2\ , R_i^/ = R_i - y_{i1} R_2^/\ , i = 1,3$									
a_3	0	9	0	-9/2	1	½		0	---
a_1	1	1	1	-1/2	0	-1/2		0	---

$a_6{}^*$	-*M*	9	0	$\frac{3^*}{2}$	0	½		1	$\frac{9}{3/2} = 6^*$
$z_j - c_j$			0	$-\frac{3}{2}M$ $-\frac{5^*}{2}$	0	$-\frac{M}{2}$ - $-\frac{1}{2}$		0	
y_{32} is the key element. $R_3^/ =$ new key row $= \frac{1}{y_{32}} R_3$, $R_i^/ = R_i - y_{i2} R_3^/$,$i = 1,2$									
a_3	0	36	0	0	1	2			
a_1	1	4	1	0	0	-1/3			
a_2	2	6	0	1	0	1/3			
$z_j - c_j$		16	0	0	0	1/3			

As $z_j - c_j \geq 0$ for all j, optimality condition is reached. The artificial vectors are all driven out from the final basis. Hence the optimal solution obtained is a B.F.S and the maximum value of z is 16 for $x_1 = 4, x_2 = 6$.

8. Solving by Big M method prove that the following L.P.P. has no F.S.
   Maximize $z = 2x_1 - x_2 + 5x_3$
   Subject to $x_1 + 2x_2 + 2x_3 \leq 2$
   $\frac{5}{2}x_1 + 3x_2 + 4x_3 = 12$
   $4x_1 + 3x_2 + 2x_3 \geq 24$, $x_j \geq 0, j = 1,2,3$.

Solution: Introducing slack, surplus and artificial variables the converted equations and the adjusted objective functions are

Maximize $z = 2x_1 - x_2 + 5x_3 + 0x_4 + 0x_5 - Mx_6 - Mx_7$

$$x_1 + 2x_2 + 2x_3 + x_4 \qquad = 2$$

$$\frac{5}{2}x_1 + 3x_2 + 4x_3 \qquad + x_6 \qquad = 12$$

$$4x_1 + 3x_2 + 2x_3 \qquad - x_5 \qquad + x_7 = 24\ , \quad x_j \geq 0, j = 1, \ldots, 7\ .$$

We now construct the simplex tables.

		c	7	-1	5	0	-M	0	-M	
Basis	c_B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7	$\min ratio = \frac{x_{Bi}}{y_{i1}}, y_{i1} > 0$
$a_4{}^*$	0	2	1^*	-1	1	0	0	0	0	$\frac{2}{1} = 2^*$
a_5	-*M*	12	5/2	9	-5	1	0	0	0	$\frac{12}{5/2} = \frac{24}{5}$
a_7	-*M*	24	4	9^*	-7	0	-1	1	1	$\frac{24}{4} = 6$
$z_j - c_j$			$-\frac{13}{2}\text{M} - 2^*$	-6M +1	-6M -5	0	0	M	0	
y_{11} is the key element. $R_1^{/}$ = new key row $= \frac{1}{y_{11}} R_1$, $R_i^{/} = R_i - y_{i1} R_1^{/}$, $i = 2,3$										
a_1	7	2	1	-1	1	0	0	0	0	
a_5	-*M*	7	0			1	0		0	
a_7	-*M*	16	0			0	1		1	
$z_j - c_j$			0	7M +5	7M -1	$\frac{13}{2}\text{M} + 2$	0	M	0	

As $z_j - c_j \geq 0$ for all j, optimality condition is reached. The artificial variables x_5 and x_7 are present at a positive level in the optimal solution. Hence the problem has no feasible solution.

Note: We need not complete the table if the optimality condition is reached.

9. Solve the L.P.P by two phase method.

Minimize $z = 3x_1 + 5x_2$

Subject to $x_1 + 2x_2 \geq 8$

$3x_1 + 2x_2 \geq 12$

$5x_1 + 6x_2 \leq 60$, $x_j \geq 0, j = 1,2$.

10. Solve the L.P.P

Maximize $z = 5x_1 + 11x_2$

Subject to $2x_1 + x_2 \leq 4$

$3x_1 + 4x_2 \geq 24$

$2x_1 - 3x_2 \geq 6$, $x_j \geq 0, j = 1,2$

by two phase method and prove that the problem has no feasible solution.

11. Solve the L.P.P

Maximize $z = 2x_1 + 5x_2$

Subject to $2x_1 + x_2 \geq 12$

$x_1 + x_2 \leq 4$, $x_1 \geq 0$, x_2 is unrestricted in sign.

CHAPTER VI

Duality Theory

Associated with every L.P.P there exists a corresponding L.P.P. The original problem is called the primal problem and the corresponding problem as the dual problem.

We will first introduce the concept of duality through an example.

Food	$F_1(x_1)$	$F_2(x_2)$	Requirement
v_1	3	2	20 units
v_2	4	3	30 units
cost	Rs. 7	Rs. 5	Per unit

Let x_1 units of F_1 and x_2 units of F_2 be required to get the minimum amount of vitamins. This is a problem of minimization. The L.P.P. is

Minimize $z = 7x_1 + 5x_2$

Subject to $3x_1 + 2x_2 \geq 20$

$4x_1 + 3x_2 \geq 30$, $x_1, x_2 \geq 0$.

Let us now consider the corresponding problem.
A dealer sells the above mentioned vitamins v_1 and v_2 separately. His problem is to fix the cost per unit of v_1 and v_2 in such a way that the price of F_1 and F_2 do not exceed the amount mentioned above. His problem is also to get a maximum amount by selling the vitamins.
Let w_1 and w_2 be the price per unit of v_1 and v_2 respectively.
Therefore the problem is
Maximize $z^* = 20w_1 + 30w_2$
Subject to $3w_1 + 4w_2 \leq 7$
$2w_1 + 3w_2 \geq 5, \quad w_1, w_2 \geq 0$.
Now if we take $A = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}, b = \begin{bmatrix} 20 \\ 30 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, c = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$,
The initial problem can be written as
Minimize $z_x = c^T x$ subject to $Ax \geq b, x \geq 0$.
Now the corresponding problem is
Maximize $z_w = b^T w$ subject to $A^T w \leq c, w \geq 0$.
The above is an example of primal-dual problem. Generally the initial problem is called the primal problem and the corresponding problem as the dual problem.
Standard form of primal
A L.P.P is said to be in standard form if
(i) All constraints involve the sign $\leq$ in a problem of maximization, or
(ii) All constraints involve the sign $\geq$ in a problem of minimization.

Given a L.P.P, we write it in the standard form as follows

Maximize $z_x = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Subject to $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$

……………………………………

$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m$, $x_j \geq 0$ for all .

Here the constants $b_i, i = 1, \ldots, m$ and $c_j, i = 1, \ldots, n$ are unrestricted in sign.

The corresponding dual problem is

Minimize $z_w = b_1w_1 + b_2w_2 + \cdots + b_mw_m$

Subject to $a_{11}w_1 + a_{21}w_2 + \cdots + a_{1n}w_m \geq c_1$

$a_{12}w_1 + a_{22}w_2 + \cdots + a_{2n}w_m \geq c_2$

……………………………………

$a_{1m}w_1 + a_{2m}w_2 + \cdots + a_{nm}w_m \geq c_n$, $w_i \geq 0$ for all , where $w = [w_1 \ w_2 \dots \dots \ w_m]^T$ is an m component dual variable vector.

Putting $A = (a_{ij})_{mxn}$, $b = (b_i)_{mx1}$, $x = (x_j)_{nx1}$, $c = (c_j)_{nx1}$, the above primal and the dual problem can be written as

Maximize $z_x = c^Tx$ subject to $Ax \leq b, x \geq 0.$
Now the corresponding problem is
Minimize $z_w = b^Tw$ subject to $A^Tw \geq c, w \geq 0.$

Theorem: Dual of the dual ts the primal itself.

Proof: Let the primal problem be Max (c^Tx) subject to $Ax \leq b, x \geq 0$ ...(1)

The dual of (1) is Min (b^Tw) subject to $A^Tw \geq c, w \geq 0$ …. (2)
(2) is equivalent to Max $(-b^Tw)$ subject to $(-A^Tw) \leq -c, w \geq 0$ ….(3)
where Min $(b^Tw) = -$Max $(-b^Tw)$ …..(4)
The dual of (3) is Min $(-c^Tx)$ subject to $(-Ax) \geq -b, x \geq 0$ ….(5)
(5) is equivalent to Max (c^Tx) subject to $Ax \leq b, x \geq 0$ which is exactly the original problem.
Hence the theorem.

From this we conclude that if either problem is considered as a primal then the other will be its dual.

Example: Write down the dual of the problem
Maximize $z = 2x_1 - 3x_2$
Subject to $x_1 - 4x_2 \leq 10$

$$-x_1 + \ x_2 \leq \ 3$$
$$-x_1 - 3x_2 \ \geq \ 4 \ , \ \ x_j \geq 0, j = 1,2 \ .$$

Solution: Rewriting the problem with all ≤ type constraints we have

Maximize $z = 2x_1 - 3x_2$

Subject to $x_1 - 4x_2 \leq \ 10$

$$-x_1 + \ x_2 \leq \ 3$$
$$x_1 + 3x_2 \ \leq -4 \ , \ \ x_j \geq 0, j = 1,2$$

which in the standard form is

Max $[2 \quad -3]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ subject to $\begin{bmatrix} 1 & -4 \\ -1 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 10 \\ 3 \\ -4 \end{bmatrix}, x_j \geq 0, j = 1,2$.

Therefore the dual of the problem is

Min $z_w = [10 \quad 3 \quad -4]\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ subject to $\begin{bmatrix} 1 & -1 & 1 \\ -4 & 1 & 3 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \geq \begin{bmatrix} 2 \\ -3 \end{bmatrix},$

$w_j \geq 0, j = 1,2,3$

Or the dual problem is Min $z_w = 10w_1 + 3w_2 - 4w_3$

Subject to $w_1 - w_2 + \ w_3 \geq \ 2$

$$-4w_1 + w_2 + 3w_3 \geq -3, w_j \geq 0, j = 1,2,3$$

Weak Duality Theorem

Theorem: If x_0 be any F.S to the primal Max $z_x = c^T x$ subject to

$Ax \leq b, x \geq 0$, and w_0 be any F.S to the dual problem

Min $z_w = b^T w$ subject to $A^T w \geq c, w \geq 0$, then $c^T x_0 \leq b^T w_0$.

Proof: We have for any F.S w_0 of the dual,

$A^T w_0 \geq c$ or, $(A^T w_0)^T \geq c^T$ or, $w_0^T A \geq c^T$. ….. (1)

x_0 be any F.S to the primal. Post multiplying (1) by x_0 we have

$(w_0^T A)x_0 \geq c^T x_0$, [since $x_0 \geq 0$]

or, $w_0^T b \geq c^T x_0$, [since $Ax_0 \leq b$]

or, $b^T w_0 \geq c^T x_0 \ \Rightarrow \ c^T x_0 \leq b^T w_0$ (as $w_0^T b$ is a scalar)

Hence the theorem is proved.

Note: If x_0 and w_0 be the optimal feasible solutions of the primal and the dual respectively then $max\ z_x \leq \min z_w$

Theorem: If x^* and w^* be any two feasible solutions of the primal max $z_x = c^T x$ subject to $Ax \leq b, x \geq 0$, and the corresponding dual, min $z_w = b^T w$ subject to $A^T w \geq c, w \geq 0$ respectively and $c^T x^* = b^T w^*$, then x^* and w^* are the optimal feasible solutions of the primal and the dual respectively.

Proof: From the previous theorem, for any two F.S. x_0 and w_0 of the primal and the dual $c^T x_0 \leq b^T w^*$ [as w^* is a F.S. of the dual]

or, $c^T x_0 \leq b^T w^* = c^T x^* \Rightarrow c^T x_0 \leq c^T x^*$,

from which we get $\max(c^T x) = c^T x^*$ which implies that x^* is an optimal feasible solution of the primal.

In the same way we can prove that $\min z_w = \min b^T w = b^T w^*$, that is, w^* is an optimal feasible solution of the dual.

Fundamental Duality Theorem

Theorem: (a) If either the primal, Max $z_x = c^T x$ subject to $Ax \leq b, x \geq 0$, or the dual, Min $z_w = b^T w$ subject to $A^T w \geq c, w \geq 0$ has a finite optimal solution, then the other problem will also have a finite optimal solution. Also the optimal values of the objective functions in both the problems will be the same, that is Max $z_x =$ Min z_w .

Proof: We first assume that the primal has an optimal feasible solution which has been obtained by simplex method. Let us convert the constraints of the primal in the following form

$$Ax + I_m x_s = b, x \geq 0, x_s \geq 0,$$

Where x_s is a set of m slack variables and I_m is a unit matrix of order m, b is unrestricted in sign as in the original problem. We assume that an optimal solution is obtained without having to make each component of the requirement vector b non negative.

Let x_B be the optimal feasible solution of the primal problem corresponding to the final basis B and let c_B be the associated cost vector. Therefore $x_B = B^{-1}b$

And the corresponding optimal value of the objective function is
$z_x = \max({c_B}^T x_B) = {c_B}^T(B^{-1}b)$.

Since x_B is optimal, we have $z_j - c_j \geq 0$ (in a maximization problem) for all j in the final table. Thus

${c_B}^T y_j - c_j \geq 0$, [y_j is the jth column vector in the final table]

or, ${c_B}^T B^{-1} a_j \geq c_j$

or, ${c_B}^T B^{-1}(a_1\ a_2 \dots a_n\ e_1\ e_2 \dots e_m) \geq (c_1\ c_2 \dots c_n\ 0\ 0 \dots 0)$

(as $e_1, e_2, \dots, e_m$ are slack vectors and cost component of each one is 0)

or, ${c_B}^T B^{-1}(A, I_m) \geq (c, 0)$

Equating we get, ${c_B}^T B^{-1} A \geq c$ and ${c_B}^T B^{-1} I_m \geq 0$. …….. (1)

Putting ${c_B}^T B^{-1} = w_0^T \geq 0$ where $w_0 = (w_1\ w_2 \dots w_m)^T$, we get from (1),

$w_0^T A \geq c^T$ and $A^T w_0 = (w_0^T A)^T \geq (c^T)^T = c$ which indicates that w_0 is a feasible solution to the dual problem.

Now we have to show that w_0 is also an optimal solution to the dual problem.

$\widehat{z_w} = b^T w_0 = (b^T w_0)^T$ (as $b^T w_0$ is a scalar)

$= w_0^T b = ({c_B}^T B^{-1})b = {c_B}^T(B^{-1}b) = {c_B}^T x_B = \max z_x$
Hence w_0 is an optimal solution to the dual problem and

$\widehat{z_w} = \min z_w = \max z_x$.

Similarly, starting with the finite optimal value of the dual problem, if it exists, we can prove that primal also has an optimal value of the objective function and $\max z_x = \min z_w$.

The above theorem can be stated in an alternative way as follows:

Theorem: A feasible solution x^* to a primal maximization problem with objective function $c^T x$, will be optimal, if and only if, there exists a feasible solution w^* to the dual minimization problem with the objective function $b^T w$ such that $c^T x^* = b^T w^*$.

The proof is almost exactly the same as the above theorem.

Theorem: (b) If either of the primal or the dual has unbounded solution then the other will have no feasible solution.

Proof: Let us assume that the primal has an unbounded solution. If the dual problem has a finite optimal solution, then the primal will also have a finite optimal solution, which is a contradiction. We now prove that the dual has no feasible solution.

When the primal objective function is unbounded,

we have $\max z_x = \max c^T x \to \infty$ and since for any feasible solution w of the dual, $b^T w \geq \max z_x = \max c^T x \to \infty$ for all feasible solutions w of the dual, which indicates that there is no feasible w whose components are finite. Hence we can conclude that the dual has no feasible solution.

Note: Converse of this theorem is not necessarily true.

Examples

1. Solve the following problem by solving its dual using simplex method.

Min $z = 3x_1 + x_2$

Subject to $2x_1 + x_2 \geq 14$

$x_1 - x_2 \geq 4 , x_1, x_2 \geq 0$

Solution: The dual of the problem is

Max $z_w = 14w_1 + 4w_2$

Subject to $2w_1 + w_2 \leq 3$

$w_1 - w_2 \leq 1 , w_1, w_2 \geq 0$

Now we solve the dual problem by the simplex method.

Simplex tables

		C	5	2	2	0	
Basis	c_B	B	a_1	a_2	a_3	a_4	Min ratio= $\frac{x_{Bi}}{y_{i1}}, y_{i1} > 0$
a_3	0	3	2	1	1	0	$\frac{3}{2}$
$a_4{}^*$	0	1	1^*	-1	0	1	$\frac{1}{1} = 1^*$
$z_j - c_j$		0	-14^*	-4	0	0	
y_{21} is the key element. $R_2^{/}$ = new key row $= \frac{1}{y_{21}} R_2$, $R_1^{/} = R_1 - y_{12} R_2^{/}$							
$a_3{}^*$	0	1	0	3^*	1	-2	$\frac{1}{3}$
a_1	14	1	1	-1	0	1	$--$
$z_j - c_j$		14	0	-18^*	0	14	
y_{12} is the key element. $R_1^{/}$ = new key row $= \frac{1}{y_{12}} R_1$, $R_2^{/} = R_2 - y_{22} R_1^{/}$							
a_2	4	1/3	0	1	1/3	-2/3	
a_1	14	4/3	1	0	1/3	1/3	
$z_j - c_j$		20	0	0	6	2	

Here all $z_j - c_j \geq 0$. So the solution is optimal.

Max $z_w = 20$ at $w_1 = \frac{4}{3}, w_2 = 1/3$.

Now $z_3 - c_3 = 6$ and $z_4 - c_4 = 2$ corresponding to the slack vectors a_3 and a_4 at the optimal stage.

Hence the primal optimal solution is $x_1 = 6, x_2 = 2$, so min z =Max $z_w = 20$ at $x_1 = 6, x_2 = 2$.

Note: Advantage of solving the dual problem is that we are able to solve the primal without using artificial variables.

2. Solve the problem by solving its dual using simplex method.

$$\text{Max } z = 3x_1 + 4x_2$$

$$\begin{aligned} \text{Subject to} \quad x_1 + \quad x_2 &\le 10 \\ 2x_1 + 3x_2 &\le 18 \\ x_1 \qquad &\le \; 8 \\ x_2 &\le \; 6 \,, x_1, x_2 \ge \; 0 \end{aligned}$$

Solution: The dual of the problem is

$$\text{Min } z_w = 10w_1 + 18w_2 + 8w_3 + 6w_4$$

$$\text{Subject to} \quad w_1 + 2w_2 + \; w_3 \qquad \ge 3$$

$$w_1 + 3w_2 \qquad + w_4 \ge 4 \,, w_1, w_2, w_3, w_4 \ge \; 0$$

Now we solve the dual problem by the simplex method.

After introducing surplus variables (as the constraints are ≥ type) we see that identity matrix is already present in the coefficient matrix. So we do not need to add artificial variables. Then we change the problem to a maximization problem as

$$\text{Max } (-z_w) = -10w_1 - 18w_2 - 8w_3 - 6w_4$$

$$\text{Subject to} \quad w_1 + 2w_2 + \; w_3 \qquad - w_5 \qquad = 3$$

$$w_1 + 3w_2 \qquad + w_4 \qquad - w_6 = 4 \,, w_1, \dots, w_6 \ge \; 0$$

Simplex tables

		c	-10	-18	-8	-6	0	0	
Basis	c_B	b	a_1	a_2	a_3	a_4	a_5	a_6	Min ratio= $\frac{x_{Bi}}{y_{i1}}, y_{i1} > 0$
a_3	-8	3	1	2	1	0	-1	0	$\frac{3}{2}$ =10
$a_4{}^*$	-6	4	1	3^*	0	1	0	-1	$\frac{4^*}{3}$
$z_j - c_j$		-48	-4	-16^*	0	0	8	6	
y_{22} is the key element. $R_2^{/}$ = new key row $= \frac{1}{y_{22}} R_2$, $R_1^{/} = R_1 - y_{i2} R_2^{/}$									
a_3	-8	1/3							

a_2	-18	4/3						
$z_j - c_j$		-80/3	4/3	0	0	16/3	8	2/3

Here all $z_j - c_j \geq 0$. So the solution is optimal.

Min $z_w = -$Max $(-z_w) = 80/3$ at $w_1 = 0, w_2 = \frac{4}{3}, w_3 = \frac{1}{3}, w_4 = 0$. Thus using the property of duality theory, Max $z =$ Min $z_w = 80/3$ at $x_1 = 8, x_2 = \frac{2}{3}$ which are the $z_j - c_j$ values corresponding to the surplus vectors $a_5(-e_1)$ and $a_6(-e_2)$ respectively.

3. Solve the following primal problem by solving its dual.
   Min $z = 10x_1 + 2x_2$
   Subject to $x_1 + 2x_2 + 2x_3 \geq 1$
   $x_1 - 2x_3 \geq -1$
   $x_1 - x_2 + 3x_3 \geq 3$, $x_1, x_2, x_3 \geq 0$

Solution: The dual of the above problem is Max $z_w = w_1 - w_2 + 3w_3$

Subject to $w_1 + w_2 + w_3 \leq 10$

$2w_1 - w_3 \leq 2$

$2w_1 - 2w_2 + 3w_3 \leq 0,\ w_1, w_2, w_3 \geq 0$

After introducing slack variables the converted problem is

Max $z_w = w_1 - w_2 + 3w_3 + 0.w_4 + 0.w_5 + 0.w_6$

Subject to $w_1 + w_2 + w_3 + w_4 = 10$
$2w_1 - w_3 + w_5 = 2$
$2w_1 - 2w_2 + 3w_3 + w_6 = 0$, $w_j \geq 0, j = 1, \ldots, 6$

Simplex tables

	c	1	-1	3	0	0	0	
Basis c_B	b	a_1	a_2	a_3	a_4	a_5	a_6	Min ratio= $\frac{x_{Bi}}{y_{i1}}, y_{i1} > 0$

a_4	0	10	1	1	1	1	0	0	$\frac{10}{1}$ =10
a_5	0	2	2	0	-1	0	1	0	… …
$a_6{}^*$	0	0	2	-2	3*	0	0	1	$\frac{0}{3} = 0^*$
$z_j - c_j$		0	-1	1	-3*	0	0	0	
y_{33} is the key element. $R_3^{/}$ = new key row $= \frac{1}{y_{33}} R_3$, $R_i^{/} = R_i - y_{i3} R_3^{/}$, $i = 1,2$									
$a_4{}^*$	0	10	1/3	$\frac{5^*}{3}$	0	1	0	-1/3	$\frac{10}{5/3} = 6^*$
a_5	0	2	8/3	-2/3	0	0	1	1/3	---
a_3	3	0	2/3	-2/3	1	0	0	1/3	---
$z_j - c_j$		0	1	-1^*	0	0	0	1	
y_{13} is the key element. $R_1^{/}$ = new key row $= \frac{1}{y_{13}} R_{13}$, $R_i^{/} = R_i - y_{i3} R_1^{/}$,$i = 2,3$									
a_2	6								
a_5	6								
a_3	4								
$z_j - c_j$		6	6/5	0	0	3/5	0	4/5	

As $z_j - c_j \geq 0$ for all j, optimality condition is reached. Hence the optimal solution obtained is a B.F.S and the maximum value of z_w is 6 for $w_1 = 0, w_2 = 6, w_3 = 4$.

Therefore min $z_x = 6$. Now $z_4 - c_4 = \frac{3}{5}, z_5 - c_5 = 0, z_6 - c_6 = 4/5$.

Thus $min\ z_w = 6$ for $x_1 = 3/5, x_2 = 0, x_3 = 4/5$.

CHAPTER VII

TRANSPORTATION AND ASSIGNMENT PROBLEM

Transportation problem (T.P) is a particular type of linear programming problem. Here, a particular commodity which is stored at different warehouses (origins) is to be transported to different distribution centres (destinations) in such a way that the transportation cost is minimum.

Let us consider an example where there are m origins O_i with the quantity available at each O_i be a_i , $i = 1,2, \dots, m$ and n destinations D_j with the quantities required, i.e. the demand at each D_j be b_j , $j = 1,2, \dots, n$.

We make an assumption $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = M$. This assumption is not restrictive.

If in a particular problem this condition is satisfied, it is called a balanced transportation problem.

If the condition is $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$, then it is called an unbalanced transportation problem.

We shall first discuss balanced transportation problems.

Destinations

		D_1	D_2	...	D_j	...	D_n		
	O_1	x_{11}	x_{12}	...	x_{1j}	...	x_{1n}	a_1	
	O_2	x_{21}	x_{22}	...	x_{2j}	...	x_{2n}	a_2	
	...	...	...	...	...	...	...	...	
origin	O_i	x_{i1}	x_{i2}	...	x_{ij}	...	x_{in}	a_i	capacities
	...	...	...	...	...	...	...	...	
	O_m	x_{m1}	x_{m2}	...	x_{mj}	...	x_{mn}	a_m	
		b_1	b_2	...	b_j	...	b_n		

Demands

In the above table, x_{ij} denotes the number of units transported from the ith origin to the jth destination.

Let c_{ij} denote the cost of transporting each unit from the ith origin to the jth destination. In general, $c_{ij} \geq 0$ for all $, j$.

The problem is to determine the quantities $x_{ij}, i = 1,2, \dots, m, j = 1,2, \dots, n$, which is to be transported from the ith origin to the jth destination such that the transportation cost is minimum subject to the condition $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

Mathematically the problem can be written as

$\min z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

subject to the constraints $\sum_{j=1}^{n} x_{ij} = a_i , i = 1,2, \dots, m$

$$\sum_{i=1}^{m} x_{ij} = b_j , j = 1,2, \dots, n$$

and $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

Clearly $x_{ij} \geq 0$ for all i, j.

We state a few theorems without proof.

Theorem: There exists a feasible solution to each T.P. which is given by $x_{ij} = \frac{a_i b_j}{M}$, $i = 1,2, \dots, m, j = 1,2, \dots, n$, where $M = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

Theorem: In each T.P. there exists at least one B.F.S. which makes the objective function a minimum.

Theorem: In a balanced T.P. having m origins and n destinations $(m, n \geq 2)$ the exact number of basic variables is $m + n - 1$.

T.P. is a special case of a L.P.P. , therefore it can be solved by using simplex method. But the method is not suitable for solving a T.P. A specially designed table called the transportation table is used to solve a T.P.

In this table there are mn squares or rectangles arranged in m rows and n columns. Each squares or rectangle is called a cell. The cell which is in the ith row and the jth column is called the $(i, j)th$ cell. Each cost component c_{ij} is displayed at the bottom right corner of the corresponding cell. A component x_{ij} (if $\neq 0$) of a feasible solution is displayed inside a square at the top left hand corner of the cell

(i, j) . The capacities of the origins and demands of the destinations are listed in the outer column and the outer row respectively as given in the table below.

Transportation Table

Destinations

		D_1	D_2	...	D_j	...	D_n		
	O_1	c_{11}	c_{12}	...	c_{1j}	...	c_{1n}	a_1	
	O_2	c_{21}	c_{22}	...	c_{2j}	...	c_{2n}	a_2	
	...	...	...	...	...	...	...	...	
Origin	O_i	c_{i1}	c_{i1}	...	c_{ij}	...	c_{in}	a_i	Capacities
	...	...	...	...	...	...	...	...	
	O_m	c_{m1}	c_{m2}	...	c_{mj}	...	c_{mn}	a_m	
		b_1	b_2	...	b_j	...	b_n		

Demands

Determination of an initial B.F.S

We will discuss two methods of obtaining an initial B.F.S , (i) North-West corner rule and (ii) Vogel's Approximation method (VAM) .

(i) North-West corner rule

Step 1: Compute $\min(a_1, b_1)$. Select $x_{11} = \min(a_1, b_1)$ and allocate the value of x_{11} in the cell (1,1), i.e. the cell in the North-West corner of the transportation table.

Step 2: If $a_1 < b_1$, the capacity of the origin O_1 will be exhausted , so all other cells in the first row will be empty, but some demand remains in the destination D_1. Compute $\min(a_2, b_1 - a_1)$. Select $x_{21} = \min(a_2, b_1 - a_1)$ and allocate the value of x_{21} in the cell (2,1).

Step 3: If $b_1 - a_1 < a_2$, then the demand of D_1 is satisfied, so all the remaining cells in the first column will be empty, but some capacity remains in the origin O_2 . Compute $\min(a_2, b_1 - a_1)$. Select $x_{22} = \min(a_2 - b_1 + a_1 , b_2)$ and allocate the value of x_{22} in the cell (2,2).

Repeat the above process till the capacities of all the origins are exhausted and the demands of all the destinations are satisfied.

The feasible solution obtained by this method is always a B.F.S.

Problem: Determine an initial B.F.S. of the following transportation problem by North-West corner rule.

destinations

		D_1	D_2	D_3	D_4		
	O_1	4	6	9	5	16	
origin	O_2	2	6	4	1	12	capacities
	O_3	5	7	2	9	15	
		12	14	9	8		

demands

Solution:

	D_1	D_2	D_3	D_4	
O_1	$\boxed{12}$ 4	$\boxed{4}$ 6	9	5	16
O_2	2	$\boxed{10}$ 6	$\boxed{2}$ 4	1	12
O_3	5	7	$\boxed{7}$ 2	$\boxed{8}$ 9	15
	12	14	9	8	

(ii) Vogel’s Approximation method (VAM)

Step 1: Determine the difference between the lowest and next to lowest cost for each row and each column and display them within first brackets against the respective rows and columns.

Step 2: Find the row or column for which this difference is maximum. Let this occur at the i^th^ row. Select the lowest cost in the i^th^ row. Let it be c_{ij} . Allocate

$x_{ij} = \min(a_i, b_j)$ in the cell (i, j). If the maximum difference is not unique then select arbitrarily.

Step 3: If $a_i < b_j$, cross out the i^th^ row and diminish b_j by a_i . If $b_j < a_i$, cross out the j^th^ column and diminish a_i by b_j . If $a_i = b_j$, delete only one of the i^th^ row or the j^th^ column.

Step 4: Compute the row and the column differences for the reduced transportation table and repeat the procedure discussed above till the capacities of all the origins are exhausted and the demands of all the destinations are satisfied.

Problem: Determine an initial B.F.S. of the following transportation problem by VAM.

destinations

		D_1	D_2	D_3	D_4		
	O_1	19	30	50	10	7	
origin	O_2	70	30	40	60	9	capacities
	O_3	40	8	70	20	18	
		5	8	7	14		

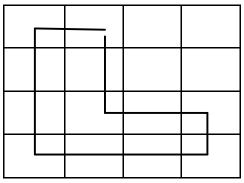
demands

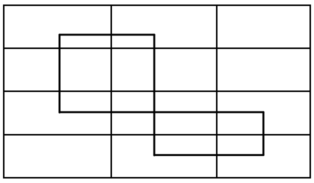
Solution:

	D_1	D_2	D_3	D_4					
O_1	$\boxed{5}$ 19	30	50	$\boxed{2}$ 10	7(9)	7(9)	2(40)	2(40)	
O_2	70	30	$\boxed{7}$ 40	$\boxed{2}$ 60	9(10)	9(20)	9(20)	9(20)	9
O_3	40	$\boxed{8}$ 8	70	$\boxed{10}$ 20	18(12)	10(20)	10(50)		
	5(21)	8(22)	7(10)	14(10)					
	5(21)		7(10)	14(10)					
			7(10)	14(10)					
			7(10)	4(50)					
			7	2					

Loops in a Transportation problem

In a transportation problem, an ordered set of four or more cells are said to form a loop (i) if and only if two consecutive cells in the ordered set lie either in the same row or in the same column and (ii) the first and the last cells in the ordered set lie either in the same row or in the same column.





These are two examples of loops.

Optimality conditions

After determining the initial B.F.S. we need to test whether the solution is optimal. Find the values of $z_{ij} - c_{ij}$ corresponding to all the non basic variables. If $z_{ij} - c_{ij} \leq 0$ for all cells corresponding to the non basic variables, the solution is optimal.

If the condition $z_{ij} - c_{ij} \leq 0$ is not satisfied for all the non basic cells, the solution is not optimal.

Determination of the net evaluation $z_{ij} - c_{ij}$ ($u\ v$ method)

The net evaluations for the non basic cells are calculated using duality theory. We give below the procedure for finding the net evaluations without proof.

We introduce $(m+n)$ quantities $u_i, v_j, i = 1,2,\ldots,m, j = 1,2,\ldots,n$ satisfying the conditions $u_i + v_j = c_{ij}$ for all the basic cells. Number of basic variables for a non degenerate solution is $m + n - 1$, so the number of equations to solve for the

$(m+n)$ quantities u_i, v_j are $+n-1$. We select arbitrarily one of u_i, v_j to be zero and then solve for the rest. Then for the non basic cells, the net evaluation $z_{ij} - c_{ij} = u_i + v_j - c_{ij}$.

If at least one $z_{ij} - c_{ij} > 0$, the solution is not optimal. As in the simplex method, our problem now is to get an optimal solution. First we have to select an entering vector and a departing vector which will move the solution towards optimality.

Determination of the departing cell, the entering cell and the value of the basic variable in the entering cell-------

If $\max_{i,j}\{z_{ij} - c_{ij}\ ,\ z_{ij} - c_{ij} > 0\} = z_{pk} - c_{pk}$, then (p,k) is the entering cell. If the maximum is not unique then select any one cell corresponding to the maximum value of the net evaluations.

If the solution is non degenerate, then it is always possible to construct a loop connecting the cell (p,k) and the set or any subset of the basic cells. Construct the loop by trial and error method and the loop is unique.

Allocate a value $\theta > 0$ in the cell (p,k) and readjust the basic variables in the ordered set of cells containing the simple loop by adding and subtracting the value θ alternately from the corresponding quantities such that all rim requirements are satisfied. Select the maximum value of θ in such a way that the readjusted values of the variables vanish at least in one cell {excluding the cell (p,k)} of the ordered set and all other variables remain non negative. The cell where the variable vanishes is the departing cell. If there are more than one such cell, select one arbitrarily as the departing cell, the remaining cells with variable values zero are kept in the basis with allocation zero. In this case the solution at the next iteration will be degenerate. The method of solving for this type of problems will be discussed later.

Construct a new transportation table with the new B.F.S. and test for optimality. Repeat the process till an optimal solution is obtained.

Problem: Determine the minimal cost of transportation for the problem given earlier where a B.F.S is obtained by VAM.

Solution: We take the initial B.F.S. as obtained earlier.

	D_1	D_2	D_3	D_4	
O_1	$\boxed{5}$ 19	30 $\boxed{-32}$	50 $\boxed{-60}$	$\boxed{2}$ 10	$u_1 = 0$
O_2	70 $\boxed{-1}$	30 $\boxed{18}$	$\boxed{7}$ 40	$\boxed{2}$ 60	$u_2 = 50$
O_3	40 $\boxed{-11}$	$\boxed{8}$ 8	70 $\boxed{-70}$	$\boxed{10}$ 20	$u_3 = 10$
	$v_1 = 19$	$v_2 = -2$	$v_3 = -10$	$v_4 = 10$	

First we have to test whether the initial B.F.S. is optimal. For that we calculate the net evaluations for all the non basic cells.

We introduce $(3 + 4)$ quantities $u_i, v_j, i = 1,2,3, j = 1,2,3,4$ satisfying the conditions $u_i + v_j = c_{ij}$ for all the basic cells. We select arbitrarily one of $u_1 = 0$. Then for the basic cell (1,1), $u_1 + v_1 = 19$ gives $v_1 = 19$.

Again for the basic cell (1,4), $u_1 + v_4 = 10$ gives $v_4 = 10$.

For the cell (2,4), $u_2 + v_4 = 60$. $v_4 = 10$ gives $u_2 = 50$.

Similarly, $u_3 + v_4 = 20$ gives $u_3 = 10$.

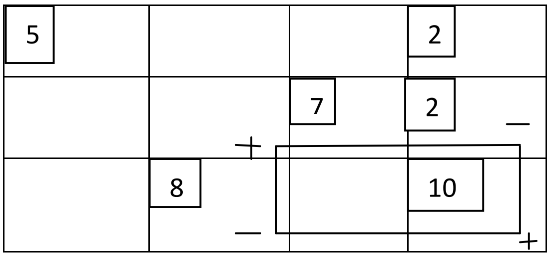
From the cells (2,3) and (3,2) we get $v_3 = -10$ and $v_2 = -2$.

So the net evaluations for the non basic cells are $u_i + v_j - c_{ij}$. We indicate the net evaluations at the bottom left hand corner of the non basic cells.

As the net evaluation of the cell (2,2) is $18 > 0$, the solution is not optimal.

To find the entering cell, calculate $\max_{i,j}(u_i + v_j - c_{ij})$ for the cells with positive net evaluation. Since only one cell (2,2) is with positive net evaluation, the entering cell is (2,2).

To find the departing cell, construct a loop with one vertex as (2,2) and all other vertices as basic cells.



To get the departing cell, we have to make one of the allocated cells empty. That is possible if we subtract minimum allocation from the cells marked with negative sign, 2 in this case, and add the same amount to allocations in the cells marked with a positive sign so that the row and column requirements are satisfied.

So the next transportation table is

$\boxed{5}$ 19	30 $\boxed{-32}$	50 $\boxed{-42}$	$\boxed{2}$ 10	$u_1 = 0$
70 $\boxed{-19}$	$\boxed{2}$ 30	$\boxed{7}$ 40	60 $\boxed{-18}$	$u_2 = 32$
40 $\boxed{-11}$	$\boxed{6}$ 8	70 $\boxed{-52}$	$\boxed{12}$ 20	$u_3 = 10$
$v_1 = 19$	$v_2 = -2$	$v_3 = 8$	$v_4 = 10$	

Here all net evaluations $z_{ij} - c_{ij} \leq 0$ for all the non basic cells. Hence the solution is optimal. The minimum cost of transportation is

$$x_{11}c_{11} + x_{14}c_{14} + x_{22}c_{22} + x_{23}c_{23} + x_{32}c_{32} + x_{34}c_{34}$$

$$= 5x19 + 2x10 + 2x30 + 7x40 + 6x8 + 12x20 = 743$$

Unbalanced Transportation Problem

In a transportation problem, if the condition is $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$, then it is called an unbalanced transportation problem. Two cases may arise.

(i) $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$

This problem can be converted into a balanced transportation problem by introducing a fictitious destination D_{n+1} with demand $b_{n+1} = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$.

The cost of the components $c_{i\,n+1} = 0\ for\ \ i = 1,2, \dots, m.$

With these assumptions, the T.P. will be a balanced one having m origins and n+1 destinations. This problem can now be solved by the previous methods.

(ii) $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j$

This problem can be converted into a balanced transportation problem by introducing a fictitious origin O_{m+1} with capacity $a_{m+1} = \sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i$.

The cost of the components $c_{m+1\,j} = 0\ for\ \ j = 1,2, \dots, n.$

With these assumptions, the T.P. will be a balanced one having $m + 1$ origins and n destinations. This problem can also be solved by the previous methods.

Problem: Solve the following unbalanced T.P. after obtaining a B.F.S. by VAM .

	D_1	D_2	D_3	D_4	
O_1	14	19	11	20	10
O_2	19	12	14	17	15
O_3	14	16	11	18	12
	8	12	16	14	

Solution: This is an unbalanced T.P. as $\sum_{j=1}^{n} b_j = 50 > \sum_{i=1}^{m} a_i = 37$. So, introducing a fictitious origin with capacity $50 - 37 = 13$ and assigning the cost of transportation from this origin to any destination as 0, we rewrite the problem as follows:

	D_1	D_2	D_3	D_4	
O_1	14	19	11	20	10
O_2	19	12	14	17	15
O_3	14	16	11	18	12
O_3	0	0	0	0	
	8	12	16	14	

Initial B.F.S. is obtained below by using VAM .

	D_1	D_2	D_3	D_4					
O_1	14	19	$\boxed{10}$ 11	20	10(3)	10(3)	10(3)		
O_2	19	$\boxed{12}$ 12	$\boxed{2}$ 14	$\boxed{1}$ 7	15(2)	15(2)	3(3)	3(3)	3(3)
O_3	$\boxed{8}$	16	$\boxed{4}$ 11	18	12(3)	12(3)	12(3)	12(2)	4(7)
O_4	0	0	0	$\boxed{13}$ 0	13(0)				
	8(14)	12(12)	16(11)	14(17)					
	8(0)	12(4)	16(0)	1(1)					
	8(0)		16(0)	1(1)					
	8(5)		6(3)	1(1)					
			6(3)	1(1)					

The solution is a non-deg B.F.S. We find the quantities u_i , v_j and test optimality.

14 $\boxed{0}$	19 $\boxed{-10}$	$\boxed{10}$ 11	20 $\boxed{-6}$	$u_1 = -3$
19 $\boxed{-2}$	$\boxed{12}$ 12	$\boxed{2}$ 14	$\boxed{1}$ 17	$u_2 = 0$
$\boxed{8}$ 14	16 $\boxed{-7}$	$\boxed{4}$ 11	18 $\boxed{-4}$	$u_3 = -3$
0 $\boxed{0}$	0 $\boxed{-5}$	0 $\boxed{-3}$	$\boxed{13}$ 0	$u_4 = -17$
$v_1 = 17$	$v_2 = 12$	$v_3 = 14$	$v_4 = 17$	

Here all cell evaluations are less than or equal to zero. Hence the solution is optimal. As one cell evaluation is zero, there exists alternative optimal solution. One optimal solution is

Solution to a degenerate problem

Degeneracy may occur at any stage of the problem. Here we will discuss degeneracy at the initial B.F.S. and only one basic variable is zero. Even if more than one basic variable is zero, the problem can be solved similarly.

Allocate a small positive quantity ε in the cell where the basic variable is zero and readjust all basic variables in the cells such that $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = M$ is satisfied. Now solve the problem as we solve a non degenerate problem and put $\varepsilon = 0$ in the final solution.

We explain the method by an example.

Problem: Solve the following balanced transportation problem.

	D_1	D_2	D_3	
O_1	8	7	3	60
O_2	3	8	9	70
O_3	11	3	5	80
	50	80	80	

We find the initial B.F.S. by VAM which is given below.

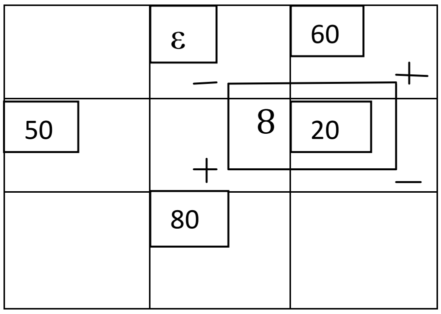
	D_1	D_2	D_3
O_1	8	7	$\boxed{60}$ 3
O_2	$\boxed{50}$ 3	8	$\boxed{20}$ 9
O_3	11	$\boxed{80}$ 3	5

This solution is degenerate as the number of allocated cells is $4 < 5 = 3 + 3 - 1 = m + n - 1$.

To resolve the degeneracy we allocate a small positive quantity ε to a cell such that a loop is not formed among some or all of the allocated cells including this new allocated cell and make them dependent. For a dependent set of cells, unique determination of u_i and v_j will not be possible. We allocate a positive quantity ε to the cell (1,2) and construct the new table and then compute u_i and v_j and the cell evaluations.

	D_1	D_2	D_3	
O_1	8 -11	ε 7	60 3	u_1 =0
O_2	50 3	8 5	20 9	$u_2 = 6$
O_3	11 -18	80 3	5 -6	$u_3 = -4$
	$v_1 = -3$	$v_2 = 7$	$v_3 = 3$	

The cell evaluation for the cell (2,2) is positive, so the solution is not optimal. We allocate maximum possible amount to the cell (2,2) and adjust the allocations in the other cells such that the allocated cells are independent, total number remain 5 and satisfy the rim requirements.



	ε —	60 +
50	8 +	20 —
	80	

	8 -11	7 -5	60+ ε 3	u_1 =-6
	50 3	ε 8	20+ ε 9	$u_2 = 0$
	11 -13	80 3	5 -1	$u_3 = -5$
	$v_1 = 3$	$v_2 = 8$	$v_3 = 9$	

Here all the cell evaluations are all negative. Hence the solution is optimal. Hence by making $\varepsilon \to 0$ we get the optimal solution as

$x_{13} = 60,\ x_{21} = 50, x_{23} = 20, x_{32} = 80$ and

the minimum cost is $60x3 + 50x3 + 20x9 + 80x3 = 750.$

Assignment Problem

Assignment problem is a particular type of transportation problem where n origins are assigned to an equal number of destinations in one to one basis such that the assignment cost (profit) is minimum (maximum).

Mathematical formulation of an assignment problem

Let x_{ij} be a variable defined by

$x_{ij} = 1$ if the ith origin is assigned to the jth destination,

$x_{ij} = 0$ the ith origin is not assigned to the jth destination.

Now assignment problem is

Optimize $z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$

Subject to $\sum_{j=1}^{n} x_{ij} = a_i = 1, i = 1,2, \dots, n$

$\sum_{i=1}^{n} x_{ij} = b_j = 1, j = 1,2, \dots, n$

and $\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} b_j = n$.

From the above it is clear that the problem is to select n cells in a nxn transportation table, only one cell in each row and each column, such that the sum of the corresponding costs (profits) be minimum (maximum). Obviously the solution obtained is a degenerate solution. To get the solution we first state a theorem.

Theorem: Given a cost or profit matrix $C = (c_{ij})_{nxn}$, if we form another matrix $C^* = (c_{ij}{}^*)_{nxn} = c_{ij} - u_i - v_j$, u_i and v_j are arbitrary chosen constants, the solution of C will be identical with that of C^*.

From this theorem we can conclude that if in an assignment problem we subtract a number from every element of a row or column of the cost (or profit) matrix. Then an assignment which is optimum for the transformed matrix will be optimum for the original matrix also.

Computational procedure for an assignment problem

We will consider a minimization problem. If it is a maximization problem, convert it to minimization by changing the cost matrix C to $-C$. Subtract the minimum cost element of each row from all other elements of the respective row. Then subtract the minimum element of each column from all other elements of that column of the resulting cost matrix (These two operations can be interchanged). A set of k zeroes $(k \geq n)$ will be obtained in the new cost matrix. Draw a minimum number of horizontal/vertical lines to cover all the zeroes. Let the number of lines be N. Then two cases may arise.

(i) $N = n$: Then the optimality condition is satisfied. We have to select a single zero in each row and each column of the table. To do this, initially cross off all zeroes which lie at the intersection of the lines. Next select a row or a column containing only one zero and cross off all zeroes of the corresponding column or row. Proceed in this way to get n zeroes. The sum of the cost components of the original cost matrix corresponding to the n zeroes of the final matrix gives the minimum cost. If the selection of n zeroes is unique, then the solution is unique, otherwise it is not unique.

(ii) $N < n$: In this case the optimality condition is not satisfied. Find the minimum element from all the uncovered elements in the last table. Subtract this element from all the uncovered elements, add to all the elements at a crossing of two lines. Keep the remaining elements unaltered. Again draw a minimum number of horizontal/vertical lines to cover all the zeroes, then repeat the procedure mentioned above.

Problem: Find the optimal assignment cost and the corresponding assignment for the following cost matrix.

	A	B	C	D	E
1	9	8	7	6	4
2	5	7	5	6	8
3	8	7	6	3	5
4	8	5	4	9	3
5	6	7	6	8	5

→

	A	B	C	D	E
1	9	8	7	6	4
2	5	7	5	6	8
3	8	7	6	3	5
4	8	5	4	9	3
5	6	7	6	8	5

	A	B	C	D	E
1	9	8	7	6	4
2	5	7	5	6	8
3	8	7	6	3	5
4	8	5	4	9	3
5	6	7	6	8	5

Subtract the minimum 4, 5, 3,3, 5 of the first, second, third, fourth, fifth rows from each element of the respective row, repeat the same for the columns.