SYLLABUS

MODULE –I (Algebra-1) (36 classes) (50 marks)

Complex Numbers[4] :De Moivre's theorem and its applications(4)

Polynomials[16]: Fundamental Theorem of Classical Algebra (statement only).Nature of roots of

an equation (surd or complex roots occur in pairs) (2).Synthetic Division(2)Statement of Descarte'

s rule of signs and its applications(2). Statement of Bolzano's theorem on continuity in case of

polynomials. Relation between roots and coefficients(2). Symmetric functions of roots(1).

Transformation of equations (4). Cardan's method of solving a cubic & Ferrari's method for a

biquadratic equation(3).

Set Theory& Relations [8]:Laws of algebra of sets &De Morgan's laws(2). Cartesian product of

sets(2)

Relations on a set. Reflexive, symmetric and transitive properties of a relation on a set(2).

Equivalence relations, equivalence class& partitions- illustrative discussions (2).

Mappings[8]: Injective and surjective mapping(2). Composition of mappings—concept only(1).

Identity and inverse mappings(2). Binary operations on a set, Identity element & Inverse

elements(3)

CLASSICAL ALGEBRA

CHAPTER I

COMPLEX NUMBERS: DE' MOIVRE'S THEOREM

A complex number z is an ordered pair of real numbers (a,b): a is called the real part of z, denoted by Re z and b is called the imaginary part of z, denoted by Im z. If Re z=0, then z is called purely imaginary; if Im z =0, then z is called real. On the set C of all complex numbers, the relation of equality and the operations of addition and multiplication are defined as follows:

(a,b)=(c,d) iff a=b and c=d, (a,b)+(c,d)=(a+c,b+d), (a,b).(c,d)= (ac-bd,ad+bc)

The set C of all complex numbers under the operations of addition and multiplication as defined above satisfies following properties:

- For $z_1,z_2,z_3 \in C$, (1) $(z_1+z_2)+z_3=z_1+(z_2+z_3)$(associativity), (2)$z_1+(0,0)=z_1$, (3) for $z=(a,b) \in C$, there exists $-z=(-a,-b) \in C$ such that $(-z)+z=z+(-z)=(0,0)$, (4)$z_1+z_2=z_2+z_1$.
- For $z_1,z_2,z_3 \in C$, (1) $(z_1.\ z_2).z_3=z_1.(z_2.z_3)$(associativity), (2)$z_1.(1,0)=z_1$, (3) for $z=(a,b) \in C, z \neq (0,0)$, there exists $\frac{1}{z} \in C$ such that $z.\frac{1}{z}=\frac{1}{z}.z=1$, (4)$z_1.z_2=z_2.z_1$.
- For $z_1,z_2,z_3 \in C$, $z_1.(z_2+z_3)=(z_1.z_2)+(z_1.z_3)$.

Few Observations

(1) Denoting the complex number (0,1) by i and identifying a real complex number (a,0) with the real number a, we see z=(a,b)=(a,0)+(0,b)=(a,0)+(0,1)(b,0) can be written as z=a+ib.

(2) For two real numbers a,b , $a^2+b^2=0$ implies a=0=b; same conclusion need not follow for two complex numbers, for example, $1^2+i^2=0$ but $1 \equiv (1,0) \neq (0,0) \equiv 0$ and $i=(0,1) \neq (0,0)$ ($\equiv$ denotes identification of a real complex number with the corresponding real number).

(3) For two complex numbers z_1,z_2, $z_1z_2=0$ implies $z_1=0$ or $z_2=0$.

(4) $i^2=(0,1)(0,1)=(-1,0) \equiv -1$.

(5) Just as real numbers are represented as points on a line, complex numbers can be represented as points on a plane: $z=(a,b) \leftrightarrow P: (a,b)$. The line containing points representing the real complex numbers (a,0), a real, is called the real axis and the line containing points representing purely imaginary complex numbers $(0,b) \equiv ib$ is called the imaginary axis. The plane on which the representation is made is called Gaussian Plane or Argand Plane.

Definition 1.1 Let $z=(a,b) \equiv a+ib$. The conjugate of z, denoted by $\bar{z}$, is (a,-b) $\equiv$a-ib.

Geometrically, the point (representing) $\bar{z}$ is the reflection of the point (representing) z in the real axis. The conjugate operation satisfies the following properties:

(1) $\bar{\bar{z}}$=z , (2) $\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$, (3) $\overline{z_1 z_2} = \bar{z_1} + \bar{z_2}$, (4)$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z_1}}{\bar{z_2}}$, (4) z+$\bar{z}$=2 Rez, z-$\bar{z}$=2i Im(z)

Definition 1.2 Let z=(a,b) ≡a+ib. The modulus of z, written as $|z|$, is defined as $\sqrt{a^2 + b^2}$.

Geometrically, $|z|$ represents the distance of the point representing z from the origin (representing complex number (0,0) ≡0+i0). More generally, $|z_1 - z_2|$ represents the distance between the points z_1 and z_2. The modulus operation satisfies the following properties:

(1) $|z_1 + z_2| \leq |z_1| + |z_2|$, (2) $|z_1 . z_2| = |z_1| . |z_2|$ (3) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
(4) $\big||z_1| - |z_2|\big| \leq |z_1 - z_2|$

GEOMETRICAL REPRESENTATION OF COMPLEX NUMBERS: THE ARGAND PLANE

Let z=a+ib be a complex number. In the Argand plane, z is represented by the point whose Cartesian co-ordinates is (a,b) referred to two perpendicular lines as axes, the first co-ordinate axis is called the real axis and the second the imaginary axis. Taking the origin as the pole and the real axis as the initial line, let (r,θ) be the polar co-ordinates of the point (a,b). Then a=r cos θ, b=r sin θ. Also r=$\sqrt{a^2 + b^2}$=$|z|$. Thus z=a+ib=$|z|$(cos θ+isin θ): this is called modulus-amplitude form of z. For a given z≠0, there exist infinitely many values of θ differing from one another by an integral multiple of 2π: the collection of all such values of θ for a given z≠0 is denoted by Arg z or Amp z. The principal value of Arg z , denoted by arg z or amp z, is defined to be the angle θ from the collection Arg z that satisfies the condition -$\pi < \theta \leq \pi$. Thus, Arg z={arg z+2nπ: n an integer}. arg z satisfies following properties:

(1) arg($z_1 z_2$)=argz_1+argz_2+2kπ, where k is a suitable integer from the set{-1,0,1] such that -$\pi <$ argz_1+argz_2+2k$\pi \leq \pi$,

(2) arg$\left(\frac{z_1}{z_2}\right)$ = argz_1-argz_2+2kπ, where k is a suitable integer from the set{-1,0,1] such that -$\pi <$ argz_1-argz_2+2k$\pi \leq \pi$.

Note An argument of a complex number z=a+ib is to be determined from the relations cos θ=a/$|z|$, sin θ= b/$|z|$ simultaneously and not from the single relation tan θ=b/a.

Example1.1 Find arg z where z=1+i tan$\frac{3\pi}{5}$.

»Let $1+i\tan\frac{3\pi}{5}$=r(cos θ+i sin θ). Then r^2= $\sec^2\frac{3\pi}{5}$. Thus r = - sec$\frac{3\pi}{5}$ > 0. Thus cos θ=- cos$\frac{3\pi}{5}$, sin θ=-sin $\frac{3\pi}{5}$. Hence $\theta=\pi+\frac{3\pi}{5}$. Since $\theta>\pi$, arg z= θ-2 π=-$\frac{2\pi}{5}$.

Geometrical representation of operations on complex numbers:

Addition Let P and Q represent the complex numbers $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$ on the Argand Plane respectively. It can be shown that the fourth vertex R of the parallelogram OPRQ represents the sum z_1+z_2 of z_1 and z_2.

Product Let $z_1=|z_1|(\cos\theta_1+i\,sin\,\theta_1)$ and $z_2=|z_2|(\cos\theta_2+i\,sin\,\theta_2)$ where $-\pi<\theta_1,\theta_2\leq\pi$. Thus $z_1z_2=|z_1||z_2|\{\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\}$. Hence the point representing z_1z_2 is obtained by rotating line segment OP{ where P represents z_1} through arg z_2 and then dilating the resulting line segment by a factor of $|z_2|$. In particular , multiplying a complex number by i=$\cos\left(\frac{\pi}{2}\right)+isin\left(\frac{\pi}{2}\right)$ geometrically means rotating the line segment by $\frac{\pi}{2}$.

Theorem 1.1 (De Moivre's Theorem) If n is an integer and θ is any real number, then $(\cos\theta+i\sin\theta)^n$= cos n θ+i sin n θ. If n=$\frac{p}{q}$, q natural, p integer, $|p|$and q are realtively prime, θ is any real number, then $(\cos\theta+i\sin\theta)^n$ has q number of values, one of which is cos n θ+i sin n θ.

Proof: Case1: Let n be a positive integer.

Result holds for n=1: $(\cos\theta+i\sin\theta)^1$= cos 1 θ+i sin 1 θ. Assume result holds for some positive integer k: $(\cos\theta+i\sin\theta)^k$= cos k θ+i sin k θ.Then $(\cos\theta+i\sin\theta)^{k+1}=(\cos\theta+i\sin\theta)^k(\cos\theta+i\sin\theta)$=(cos k θ+i sin k θ)(cos θ+i sin θ)= cos(k+1) θ+isin(k+1) θ. Hence result holds by mathematical induction.

Case 2: Let n be a negative integer, say, n=-m, m natural.

$(\cos\theta+i\sin\theta)^n=(\cos\theta+i\sin\theta)^{-m}=\frac{1}{(cos\theta+isin\theta)^m}=\frac{1}{\cos m\theta+i\,sin\,m\theta}$ (by case 1) = cos mθ-isin mθ=cos(-m) θ+isin(-m) θ= cos n θ+i sin n θ.

Case3: n=0: proof obvious.

Case 4: Let n=$\frac{p}{q}$, q natural, p integer, $|p|$and q are realtively prime.

Let $(\cos\theta+i\,sin\theta)^{\frac{p}{q}}$= cos φ+i sin φ. Then $(\cos\theta+i\,sin\theta)^p$= (cos φ+i sin $\varphi)^q$. Thus cos p θ+isin p θ= cos q φ+i sin q φ. Thus q φ=2kπ+p θ, that is, $\varphi=\frac{2k\pi+p\,\theta}{q}$. Hence $(\cos\theta+i\,sin\theta)^{\frac{p}{q}}$= cos($\frac{2k\pi+p\,\theta}{q}$)+isin($\frac{2k\pi+p\,\theta}{q}$), where k=0,1,…,q-1 are the distinct q values.

Some Applications of De' Moivre's Theorem

(1) Expansion of cos n θ, sin n θ and tan n θ where n is natural and θ is real.
Cos n θ+i sin n θ=(cos θ+isin θ)^n=cos^n θ+${}^n_1C$ cos^{n-1} θisin θ+${}^n_2C$ cos^{n-2} θ $i^2 sin^2\theta$+…+i^{n}sin^n θ= (cos^n θ-${}^n_2C$ cos^{n-2} θ $sin^2\theta$+…)+i(${}^n_1C$ cos^{n-1} θsin θ-${}^n_3C$ cos^{n-3} $\theta sin^3\theta$+…). Equating real and imaginary parts, cos n θ= cos^n θ-${}^n_2C$ cos^{n-2} θ $sin^2\theta$+… and sin n θ=${}^n_1C$ cos^{n-1} θsin θ-${}^n_3C$ cos^{n-3} $\theta sin^3\theta$+…

(2) Expansion of cos^n θ and sin^n θ in a series of multiples of θ where n is natural and θ is real.
Let x = cos $\theta + isin\theta$. Then x^n=cos n θ+isin n θ, x^{-n}= cos n θ-isin n θ. Thus (2 cos θ)^n=(x+$\frac{1}{x}$)^n

=(x^n+$\frac{1}{x^n}$) + ${}^n_1C$(x^{n-2}+$\frac{1}{x^{n-2}}$)+…=2 cos n θ+${}^n_1C$(2 cos(n-2) θ)+…

Similarly, expansion of sin^n θ in terms of multiple angle can be derived.

(3) Finding n th roots of unity
To find z satisfying z^n=1=cos(2kπ)+isin(2kπ), where k is an integer. Thus z=[cos(2kπ)+isin(2kπ)]$^{1/n}$=cos($\frac{2k\pi}{n}$) $+ isin\left(\frac{2k\pi}{n}\right)$, k=0,1,…,n-1; replacing k by any integer gives rise to a complex number in the set A={ cos($\frac{2k\pi}{n}$) $+ isin\left(\frac{2k\pi}{n}\right)$/ k=0,1,…,n-1}. Thus A is the set of all nth roots of unity.

Example1.2 Solve x^6+x^5+x^4+x^3+x^2+x+1=0

»We have the identity x^6+x^5+x^4+x^3+x^2+x+1=$\frac{x^7-1}{x-1}$. Roots of x^7-1=0 are cos$\frac{2k\pi}{7} + isin\frac{2k\pi}{7}$, k=0,1,…,6. Putting k=0, we obtain root of x-1=0. Thus the roots of given equation are cos$\frac{2k\pi}{7} + isin\frac{2k\pi}{7}$, k=1,…,6.

Example1.3 Prove that the sum of 99th powers of all the roots of x^7-1=0 is zero.

»The roots of x^7-1=0 are {1,α, α^2,…, α^6}, where α=cos$\frac{2\pi}{7}$+isin$\frac{2\pi}{7}$. Thus sum of 99th powers of the roots is 1+ α^{99}+$(\alpha^2)^{99} + \cdots + (\alpha^6)^{99}$ =1+ α^{99}+$(\alpha^{99})^2 + \cdots + (\alpha^{99})^6$=$\frac{1-\alpha^{99.7}}{1-\alpha^{99}}$=0, since $\alpha^{99.7}$=1 and $\alpha^{99} \neq 1$.

Example1.4 If $|z_1| = |z_2|$ and arg z_1+argz_2=0, then show that $z_1=\overline{z_2}$.

»Let $|z_1| = |z_2|$=r, arg $z_1=\theta$, then argz_2=- θ. Thus z_1=r(cos θ+isin θ)=r(cos θ-isin θ)=$\overline{z_2}$.

Example1.5 For any complex number z, show that, $|z| \geq \frac{|Re\ z|+|Im\ z|}{\sqrt{2}}$.

»Let z= x+iy. Then $2(x^2+y^2)-(x+y)^2=(x-y)^2\geq 0$. Thus $x^2+y^2\geq \frac{(x+y)^2}{2}$ and so $|z|=\sqrt{x^2+y^2} \geq \frac{|Re\ z|+|Im\ z|}{\sqrt{2}}$.

Example1.6 Prove that if the ratio $\frac{z-i}{z-1}$ is purely imaginary, then the point (representing) z lies on the circle whose centre is at the point $\frac{1}{2}(1+i)$ and radius is $\frac{1}{\sqrt{2}}$.

»Let z=x+iy. Then $\frac{z-i}{z-1} = \frac{x^2-x+y^2-y}{(x-1)^2+y^2} + i\frac{1-x-y}{(x-1)^2+y^2}$. By given condition ,$x^2 - x + y^2 - y$=0, that is, $\left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2$.Thus z lies on the circle whose centre is at the point $\frac{1}{2}(1+i)$ and radius is $\frac{1}{\sqrt{2}}$.

Example1.7 If the amplitude of the complex number $\frac{z-i}{z+1}$ is $\frac{\pi}{4}$, show that z lies on a circle in the Argand plane.

»Let z=x+iy. Then $\frac{z-i}{z+1}$=$\frac{x^2+x+y^2-y}{(x+1)^2+y^2} + i\frac{y-x-1}{(x+1)^2+y^2}$. By given condition, $\frac{y-x-1}{x^2+x+y^2-y}$=1. On simplification we get $(x+1)^2+(y-1)^2=1$. Hence z lies on the circle with centre at (-1,1) and radius 1.

Example1.8 If z and z_1 are two complex numbers such that z+z_1 and zz_1 are both real, show that, either z and z_1 are both real or $z_1=\bar{z}$.

Example1.9 If $|z_1+z_2| = |z_1-z_2|$, prove that arg z_1 and argz_2 differ by $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.

»$|z_1+z_2|^2 = |z_1-z_2|^2$ Thus $(z_1+z_2)(\overline{z_1}+\overline{z_2})$= $(z_1-z_2)(\overline{z_1}-\overline{z_2})$ Or, $z_1\overline{z_2} + \overline{z_1}z_2 = 0$**(1)**. Let z_1=r_1(cos$\theta_1 + i sin\theta_1$), z_2=r_2(cos$\theta_2 + i sin\theta_2$). From (1), cos(θ_1-θ_2)=0 proving the result.

Example1.10 If A,B,C represent complex numbers z_1,z_2,z_3 in the Argand plane and $z_1+z_2+z_3=0$ and $|z_1|=|z_2| = |z_3|$, prove that ABC is an equilateral triangle.

»$z_1+z_2=-z_3$. Hence $|z_1 + z_2|^2 = |z_3|^2$, that is, $|z_1|^2+|z_2|^2+2z_1.z_2=|z_3|^2$. By given condition, $|z_1||z_2|\cos\,\theta=|z_1|^2$, where θ is the angle between z_1 and z_2.Thus $\cos\,\theta=-\frac{1}{2}$, that is, $\theta=120^0$. Hence the corresponding angle of the triangle ABC is 60^0. Similarly other angles are 60^0.

Example1.11 If (x,y) represents a point lying on the line $3x+4y+5=0$, find the minimum value of $|x + iy|$.

Example1.12 Let z and z_1 be two complex numbers satisfying $z=\frac{1+z_1}{1-z_1}$ and $|z_1|=1$. Prove that z lies on the imaginary axis.

» $z_1=\frac{z-1}{z+1}$. By given condition, $1 = \left|\frac{z-1}{z+1}\right| = \frac{|z-1|}{|z+1|}$. If $z=x+iy$, $x=0$. Hence etc.

Example1.13 If z_1, z_2 are conjugates and z_3,z_4 are conjugates, prove that, $\arg\frac{z_1}{z_4} = \arg\frac{z_3}{z_2}$.

» Since z_1, z_2 are conjugates, arg z_1+arg z_2=0. Since z_3,z_4 are conjugates, arg z_3+arg z_4=0. Thus arg z_1 + arg z_2 = 0 = arg z_3+arg z_4. Hence arg z_1-arg z_4=arg z_3-arg z_2; thus result holds.

Example1.14 complex numbers z_1,z_2,z_3 satisfy the relation $z_1^2+z_2^2+z_3^2-z_1z_2-z_2z_3-z_3z_1=0$ iff $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$.

» $0=z_1^2+z_2^2+z_3^2-z_1z_2-z_2z_3-z_3z_1=(z_1+wz_2+w^2z_3)(z_1+w^2z_2+wz_3)$, where w stands for an imaginary cube roots of unity. If $z_1+wz_2+w^2z_3=0$, then $(z_1-z_2)=-w^2(z_3-z_2)$; hence $|z_1 - z_2|=|w^2||z_2 - z_3|=|z_2 - z_3|$.similarly other part.

Conversely, if $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$, then z_1,z_2,z_3 represent vertices of an equilateral triangle. Then $z_2-z_1=(z_3-z_1)(\cos 60^0+i\sin60^0)$, $z_1-z_2=(z_3-z_2)(\cos 60^0+i\sin60^0)$; by dividing respective sides, we get the result.

Example1.15 Prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$, for two complex numbers z_1,z_2.

»$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2})=|z_1|^2 + |z_2|^2+2\ z_1z_2$; similarly $|z_1 - z_2|^2=|z_1|^2 + |z_2|^2-2\ z_1z_2$; Adding we get the result.

Example1.16 If $\cos\alpha+\cos\beta+\cos\gamma=\sin\alpha+\sin\beta+\sin\gamma=0$, then prove that (1) $\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\cos(\alpha+\beta+\gamma)$, (2) $\sum cos^2\alpha=\sum sin^2\alpha=3/2$.

»Let $x=\cos\alpha+i\sin\alpha$, $y=\cos\beta+i\sin\beta$, $z=\cos\gamma+i\sin\gamma$. Then $x+y+z=0$. Thus $x^3+y^3+z^3=3xyz$. By De' Moivre's Theorem, $(\cos 3\alpha+\cos 3\beta+\cos 3\gamma)+i(\sin 3\alpha+\sin 3\beta+\sin 3\gamma)=3[\cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma)]$. Equating, we get result.

Let $x=\cos\alpha+i\sin\alpha$, $y=\cos\beta+i\sin\beta$, $z=\cos\gamma+i\sin\gamma$. Then $x+y+z=0$. Also $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0;$ hence $xy+yz+zx=0$. Thus $x^2+y^2+z^2=0$. By De' Moivre's Theorem, $\cos 2\alpha+\cos 2\beta+\cos 2\gamma=0$. Hence $\sum cos^2\alpha=3/2$. Using $\sin^2\alpha=1-\cos^2\alpha$, we get other part.

Example1.17 Find the roots of $z^n=(z+1)^n$, where n is a positive integer, and show that the points which represent them in the Argand plane are collinear.

Let $w=\frac{z+1}{z}.Then\ z=\frac{1}{w-1}$.Now $z^n=(z+1)^n$ implies $w^n=1$.Thus, $w=\cos\frac{2k\pi}{n}+isin\frac{2k\pi}{n}$,k=0,…,n-1.

So $z=\frac{1}{\cos\frac{2k\pi}{n}+isin\frac{2k\pi}{n}}$, k=1,…,n-1

$=-\frac{1}{2}-\frac{i}{2}cot\frac{k\pi}{n}$. Thus all points z satisfying $z^n=(z+1)^n$ lie on the line $x=-\frac{1}{2}$.

CHAPTER II

THEORY OF EQUATIONS

An expression of the form $a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$, where $a_0,a_1,\ldots,a_n$ are real or complex constants, n is a nonnegative integer and x is a variable (over real or complex numbers) is a polynomial in x. If $a_0\neq0$, the polynomial is of degree n and a_0x^n is the leading term of the polynomial. A non-zero constant a_0 is a polynomial of degree 0 while a polynomial in which the coefficients of each term is zero is said to be a zero polynomial and no degree is assigned to a zero polynomial.

Equality two polynomials $a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$ and $b_0x^n+b_1x^{n-1}+\ldots+b_{n-1}x+b_n$ are equal iff $a_0=b_0,a_1=b_1,\ldots,a_n=b_n$.

Addition Let $f(x)= a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$, $g(x)= b_0x^n+b_1x^{n-1}+\ldots+b_{n-1}x+b_n$. the sum of the polynomials f(x) and g(x) is given by

$f(x)+g(x)= a_0x^n+\ldots+a_{n-m-1}x^{m+1}+(a_{n-m}+b_0)x^m+\ldots+(a_n+b_m)$, if $m<n$

$= (a_0+b_0)\, x^n +\ldots+ (a_n+ b_n)$, if $m=n$

$= b_0x^m+\ldots+b_{m-n-1}x^{n+1} + (b_{m-n}+a_0)\, x^n+\ldots+ (b_m+a_n)$, if $m>n$.

Multiplication Let $f(x) = a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$, $g(x)= b_0x^n+b_1x^{n-1}+\ldots+b_{n-1}x+b_n$. the product of the polynomials f(x) and g(x) is given by

$f(x)g(x)=c_0x^{m+n}+c_1x^{m+n-1}+\ldots+ c_{m+n}$, where $c_i = a_0b_i+a_1b_{i-1}+\ldots+a_ib_0$. $c_0=a_0b_0\neq0$; hence degree of f(x)g(x) is m+n.

Division Algorithm Let f(x) and g(x) be two polynomials of degree n and m respectively and $n\geq m$. Then there exist two uniquely determined polynomials q(x) and r(x) satisfying $f(x)=g(x)q(x)+r(x)$, where the degree of q(x) is n-m and r(x) is either a zero polynomial or the degree of r(x) is less than m. In particular, if degree of g(x) is 1, then r(x) is a constant, identically zero or non-zero.

Theorem 1.2 (Remainder Theorem) If a polynomial f(x) is divided by x-a, then the remainder is f(a).

»Let q(x) be the quotient and r (constant)be the remainder when f is divided by x-a, then $f(x)=(x-a)q(x)+r$ is an identity. Thus $f(a)=r$.

Theorem 1.3 (Factor Theorem) If f is a polynomial, then x-a is a factor of f iff f(a)=0.

»By Remainder theorem, f(a) is the remainder when f is divided by x-a; hence, if f(a)=0, then x-a is a factor of f. Conversely, if x-a is a factor of f, then $f(x)=(x-a)g(x)$ and hence f(a)=0.

Example1.29 Find the remainder when $f(x)=4x^5+3x^3+6x^2+5$ is divided by 2x+1.

»The remainder on dividing f(x) by $x-(-\frac{1}{2})=x+\frac{1}{2}$ is $f(-\frac{1}{2})=6$. If q(x) be the quotient, then $f(x)=q(x)(x+\frac{1}{2})+6=\frac{q(x)}{2}(2x+1)+6$. Hence 6 is the remainder when f is divided by 2x+1.

Synthetic division

Synthetic division is a method of obtaining the quotient and remainder when a polynomial is divided by a first degree polynomial or by a finite product of first degree polynomials. Let $q(x)=b_0x^{n-1}+b_1x^{n-2}+\ldots+b_{n-1}$ be the quotient and R be the remainder when $f(x)=a_0x^n+a_1x^{n-1}+\ldots+a_n$ is divided by x-c. Then

$a_0x^n+a_1x^{n-1}+\ldots+a_n=(b_0x^{n-1}+b_1x^{n-2}+\ldots+b_{n-1})(x-c)+R.$

Above is an identity; equating coefficients of like powers of x, $a_0=b_0$, $a_1=b_1-cb_0$, $a_2=b_2-cb_1,\ldots,a_{n-1}=b_{n-1}-cb_{n-2},a_n=R-cb_{n-1}$.

Hence $b_0=a_0,b_1=a_1+cb_0,b_2=a_2+cb_1,\ldots,b_{n-1}=a_{n-1}+cb_{n-2},R=a_n+cb_{n-1}$.

The calculation of $b_0,b_1,\ldots,b_{n-1},R$ can be performed as follows:

$$\begin{array}{cccccc} a_0 & a_1 & a_2 & \ldots & a_{n-1} & a_n \\ & cb_0 & cb_1 & \ldots & cb_{n-2} & cb_{n-1} \\ \hline b_0 & b_1 & b_2 & \ldots & b_{n-1} & R \end{array}$$

Example1.30 Find the quotient and remainder when $2x^3-x^2+1$ is divided by 2x+1.

»

$$\begin{array}{c|cccc} -\frac{1}{2} & 2 & -1 & 0 & 1 \\ & & -1 & 1 & -½ \\ \hline & 2 & -2 & 1 & ½ \end{array}$$

Thus $2x^3-x^2+1=(x+\frac{1}{2})(2x^2-2x+1)+\frac{1}{2}=(2x+1)(x^2-x+\frac{1}{2})+\frac{1}{2}$; hence the quotient is $x^2-x+\frac{1}{2}$ and the remainder is $\frac{1}{2}$.

Example1.31 Find the quotient and remainder when $x^4-3x^3+2x^2+x-1$ is divided by x^2-4x+3.

»$x^2-4x+3=(x-1)(x-3)$. We divide $x^4-3x^3+2x^2+x-1$ by x-1 and then the obtained quotient again by x-3 by method of synthetic division. We get $x^4-3x^3+2x^2+x-1=(x-1)[x^3-2x^2+1]+0=(x-1)[(x-3)\{x^2+x+3\}+10]=(x^2-4x+3)(x^2+x+3)+10(x-1)$; hence the quotient is x^2+x+3 and the remainder is 10(x-1).

Applications of the method

(1) To express a polynomial $f(x)=a_0x^n+a_1x^{n-1}+\ldots+a_n$ as a polynomial in x-c. Let $f(x)=A_0(x-c)^n+A_1(x-c)^{n-1}+\ldots+A_n$. Then $f(x)=(x-c)[A_0(x-c)^{n-1}+A_1(x-c)^{n-2}+\ldots+A_{n-1}]+A_n$. Thus on dividing f(x) by x-c, the remainder is A_n and the remainder is $q(x)=A_0(x-c)^{n-1}+A_1(x-c)^{n-2}+\ldots+A_{n-1}$. Similarly if q(x) is

divided by x-c, the remainder is A_{n-1}. Repeating the process n times, the successive remainders give the unknowns $A_n,\ldots,A_1$ and $A_0=a_0$.

(2) Let f(x) be a polynomial in x. to express f(x+c) as a polynomial in x.
Let $f(x)=a_0x^n+a_1x^{n-1}+\ldots+a_n=A_0(x-c)^n+A_1(x-c)^{n-1}+\ldots+A_n$; by method explained above , the unknown coefficients can be found out in terms of the known coefficients $a_0,\ldots,a_n$. Now $f(x+c)=A_0x^n+\ldots+A_n$.

Example1.32 Express $f(x)=x^3-6x^2+12x-16=0$ as a polynomial in x-2 and hence solve the equation f(x)=0.

» Using method of synthetic division repeatedly, we have

2 | 1 -6 12 -16 Hence $f(x)=(x-2)^3-8=0$.

2 -8 8 Thus $x-2=2,2w,2w^2$ (w is an imaginary cube roots of unity)

---------------------------- Hence $x=4,2+2w,2+2w^2$.

1 -4 4 | -8

2 -4

----

1 -2 | 0

2

----

1 | 0

Example1.33 If f(x) is a polynomial of degree ≥2 in x and a,b are unequal, show that the remainder on dividing f(x) by (x-a)(x-b) is $\frac{(x-b)f(a)-(x-a)f(b)}{a-b}$.

» By division algorithm, let f(x)=(x-a)(x-b)q(x)+rx+s, where rx+s is the remainder. Replacing x by a and by b in turn in this identity, f(a)=ra+s, f(b)=rb+s; solving for r,s and substituting in the expression rx+s,we get required expression for remainder.

Example1.34 If x^2+px+1 be a factor of ax^3+bx+c, prove that $a^2-c^2=ab$. Show that in this case x^2+px+1 is also a factor of cx^3+bx^2+a.

»Let $ax^3+bx+c=(x^2+px+1)(ax+d)$ (*) (taking into account the coefficient of x^3). Comparing coefficients, d+ap=0,a+pd=b,d=c whence the result $a^2-c^2=ab$

follows. Replacing x by 1/x in the identity (*), we get $cx^3+bx^2+a=(x^2+px+1)(dx+a)$: this proves the second part.

If f(x) is a polynomial of degree n, then f(x)=0 is called a polynomial equation of degree n. If b is a real or complex number such that f(b)=0, then b is a root of the polynomial equation f(x)=0 or is a zero of the polynomial f(x). If $(x-b)^r$ is a factor of f(x) but $(x-b)^{r+1}$ is not a factor of f(x), then b is a root of f(x) of multiplicity r: a root of multiplicity 1 is called a simple root. Thus 2 is a simple root of $x^3-8=0$ but 2 is a root of multiplicity 3 of $(x-2)^3(x+3)=0$.

Example1.35 Show that $1-\frac{x}{1!}+\frac{x(x-1)}{2!}+\cdots+(-1)^n\frac{x(x-1)\dots(x-n+1)}{n!}=\frac{(-1)^n}{n!}(x-1)\dots(x-n)$.

»1,…,n are all zeros of the polynomial on the left; by factor theorem, each of (x-1),…,(x-n) are factors of the polynomial and hence (x-1)…(x-n) is a factor of the n th degree polynomial on the left; equating coefficient of x^n from both side, we get the constant of proportionality $\frac{(-1)^n}{n!}$ on the right.

Theorem 1.4 (Fundamental Theorem of Classical Algebra)

Every polynomial equation of degree≥1 has a root, real or complex.

Corollary A polynomial equation of degree n has exactly n roots, multiplicity of each root being taken into account.

Corollary If a polynomial f(x) of degree n vanishes for more than n distinct values of x, then f(x) =0 for all values of x.

Example1.36 $x^2-4=(x+2)(x-2)$ is an identity since it is satisfied by more than two values of x; in contrast (x-1)(x-2)=0 is an equality and not an identity.

Theorem 1.5 If b is a multiple root of the polynomial equation f(x)=0 of multiplicity r, then b is a multiple root of $f^{(1)}(x)=0$ of multiplicity r-1. Thus to find the multiple roots of a polynomial equation f(x)=0, we find the h.c.f. g(x) of the polynomials f(x) and $f^{(1)}(x)$. The roots of g(x)=0 are the multiple roots of f(x)=0.

Example1.37 Find the multiple roots of the equation $x^5+2x^4+2x^3+4x^2+x+2=0$.

Let $f(x)=x^5+2x^4+2x^3+4x^2+x+2$. Then $f^{(1)}(x)=5x^4+8x^3+6x^2+8x+1$. The h.c.f. of f and $f^{(1)}$ are obtained by method of repeated division(at each stage, we can multiply by a constant of proportionality to our convenience; that does not affect the outcome):

```
5  8  6  8  1 |  1     2     2     4     1     2
        (x 5)    5    10    10    20     5    10
              ...5...... 8...... 6...... 8...... 1............
                       2     4    12     4    10
(x5)                  10    20    60    20    50
                      10    16    12    16     2
              ..............................................
                             4    48     4    48
        (x1/4)               1    12     1    12

1   12   1   12 |   5    8    6    8    1
                    5   60    5   60
              ........................................
                        -52     1    -52     1
                        -52  -624    -52  -624
                        ........................................
                               625     0    625
      (x1/625)                   1     0      1

1   0   1 |  1    12    1    12
             1     0    1
          ..........................
                  12    0    12
```

12 0 12

……………………………..

Thus the h.c.f. is x^2+1. Thus f(x)=0 has two double roots(that is, multiple roots of multiplicity 2) i and –i.

Polynomial equations with Real Coefficients

Theorem 1.6 If a+ib is a root of multiplicity r of the polynomial equation f(x)=0 with real coefficients, then a-ib is a root of multiplicity r of f(x)=0.

Note 1+i is a root of $x^2-(1+i)x=0$ but not so is 1-i.

Example1.38 Prove that the roots of $\frac{1}{x-1}+\frac{2}{x-2}+\frac{3}{x-3}=\frac{1}{x}$ are all real.

» The given equation is $\frac{1}{x-1}+\frac{4}{x-2}+\frac{9}{x-3}=-5$ (*). Let a+ib be a root of the polynomial equation (*) with real coefficients. Then a-ib is also a root of (*).Thus $\frac{1}{(a-1)+ib}+\frac{4}{(a-2)+ib}+\frac{9}{(a-3)+ib}=-5$ and $\frac{1}{(a-1)-ib}+\frac{4}{(a-2)-ib}+\frac{9}{(a-3)-ib}=-5$. Subtracting,$-2ib[\frac{1}{(a-1)^2+b^2}+\frac{4}{(a-2)^2+b^2}+\frac{9}{(a-3)^2+b^2}]=0$ which gives b=0. Hence all roots of given equation must be real.

Example1.39 Prove that the roots of $\frac{1}{x+a_1}+\cdots+\frac{1}{x+a_n}=\frac{1}{x+b}$ are all real where $a_1,\ldots,a_n,b$ are all positive real numbers and $b>a_i$ for all i.

Example1.40 Solve the equation $f(x)=x^4+x^2-2x+6=0$, given that 1+i is a root.

» Since f(x)=0 is a polynomial equation with real coefficients, 1-i is also a root of f(x)=0. By factor theorem, $(x-1-i)(x-1+i)=x^2-2x+2$ is a factor of f(x). By division, $f(x)=(x^2-2x+2)(x^2+2x+3)$. Roots of $x^2+2x+3=0$ are $-1\pm\sqrt{2}i$. Hence the roots of f(x)=0 are $1\pm i$, $-1\pm\sqrt{2}i$.

Theorem 1.7 If $a+\sqrt{b}$ is a root of multiplicity r of the polynomial equation f(x)=0 with rational coefficients, then $a-\sqrt{b}$ is a root of multiplicity r of f(x)=0 where a,b are rational and b is not a perfect square of a rational number.

Since every polynomial with real coefficients is a continuous function from R to R, we have

Theorem 1.8 (Intermediate Value Property) Let f(x) be a polynomial with real coefficients and a,b are distinct real numbers such that f(a) and f(b) are of opposite signs. Then f(x)=0 has an odd number of roots between a and b. If f(a) and f(b) are of same sign, then there is an even number of roots of f(x)=0 between a and b.

Example1.41 Show that for all real values of a, the equation (x+3)(x+1)(x-2)(x-4)+a(x+2)(x-1)(x-3)=0 has all its roots real and simple.

»Let f(x)= (x+3)(x+1)(x-2)(x-4)+a(x+2)(x-1)(x-3). Then $\lim_{x\to-\infty} f(x)$=∞, f(-2)<0, f(1)>0, f(3)<0,$\lim_{x\to\infty} f(x)$=∞. Thus each of the intervals $(-\infty$,-2),(-2,1),(1,3),(3, ∞) contains a real root of f(x)=0. Since the equation is of degree 4, all its roots are real and simple.

Theorem 1.9 (Rolle's Theorem) Let f(x) be a polynomial with real coefficients . Between two distinct real roots of f(x)=0 ,there is at least one real root of $f^{(1)}(x)=0$.

Note

(1) Between two **consecutive** real roots of $f^{(1)}(x)=0$, there is at most one real root of f(x)=0.
(2) If all the roots of f(x)=0 be real and distinct, then all the roots of $f^{(1)}(x)=0$ are also real and distinct.

Example1.42 Show that the equation $f(x)=(x-a)^3+(x-b)^3+(x-c)^3+(x-d)^3=0$, where a,b,c,d are not all equal , has only one real root.

» Since f(x)=0 is a cubic polynomial equation with real coefficients, f(x)=0 has either one or three real roots. If α be a real multiple root of f(x)=0 with multiplicity 3, then α is also a real root of $f^{(1)}(x)=3[(x-a)^2+(x-b)^2+(x-c)^2+(x-d)^2]=0$, and hence α=a=b=c=d (since α,a,b,c,d are real), contradiction. If f(x)=0 has two distinct real roots, then in between should lie a real root of $f^{(1)}(x)=0$, contradiction since not all of a,b,c,d are equal. Hence f(x)=0 has only one real root.

Example1.43 Find the range of values of k for which the equation $f(x)=x^4+4x^3-2x^2-12x+k=0$ has four real and unequal roots.

» Roots of $f^{(1)}(x)=0$ are -3,-1,1. Since all the roots of f(x)=0 are to be real and distinct, they will be separated by the roots of $f^{(1)}(x)=0$. Now $\lim_{x\to-\infty} f(x)$=∞,f(-3)=-9+k,f(-1)=7+k,f(1)=-9+k,$\lim_{x\to\infty} f(x)$=∞. Since f(-3)<0, f(-1)>0 and f(1)<0, -7<k<9.

Example1.44 If $c_1,c_2,\ldots,c_n$ be the roots of $x^n+nax+b=0$, prove that $(c_1-c_2)(c_1-c_3)\ldots(c_1-c_n)=n(c_1^{n-1}+a)$.

» By factor theorem, $x^n+nax+b=(x-c_1)(x-c_2)\ldots(x-c_n)$.Differentiating w.r.t. x, $n(x^{n-1}+a)=(x-c_2)\ldots(x-c_n)+(x-c_1)(x-c_3)\ldots(x-c_n)+\ldots+(x-c_2)(x-c_3)\ldots(x-c_n)$. Replacing x by c_1 in this identity, we obtain the result.

Example1.45 If a is a double root of $f(x)=x^n+p_1x^{n-1}+\ldots+p_n=0$, prove that a is also a root of $p_1x^{n-1}+2p_2x^{n-2}+\ldots+np_n=0$.

» Since a is a double root of $f(x)=0$, both $f(a)=0$ and $f^{(1)}(a)=0$ hold. Thus $a^n+p_1a^{n-1}+\ldots+p_n=0$ (1) and $na^{n-1}+(n-1)p_1a^{n-2}+\ldots+p_{n-1}=0$(2). Multiplying both side of (1) by n and both side of (2) by a and subtracting, we get $p_1a^{n-1}+2p_2a^{n-2}+\ldots+np_n=0$.Hence the result.

Example1.46 Prove that the equation $f(x)=1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}=0$ cannot have a multiple root.

» If a is a multiple root of $f(x)=0$, then $1+a+\frac{a^2}{2!}+\ldots+\frac{a^n}{n!}=0$ and $1+a+\frac{a^2}{2!}+\ldots+\frac{a^{n-1}}{(n-1)!}=0$;it thus follows that $\frac{a^n}{n!}=0$, so that a=0; but 0 is not a root of given equation. Hence no multiple root.

Descartes' Rule of signs

Theorem 1.10 The number of positive roots of an equation f(x)=0 with real coefficients does not exceed the number of variations of signs in the sequence of the coefficients of f(x) and if less, it is less by an even number.

The number of negative roots of an equation f(x)=0 with real coefficients does not exceed the number of variations of signs in the sequence of the coefficients of f(-x) and if less, it is less by an even number.

Example1.47 If $f(x)=2x^3+7x^2-2x-3$, express f(x-1) as a polynomial in x. Apply Descartes' rule of signs to both the equations f(x)=0 and f(-x)=0 to determine the exact number of positive and negative roots of f(x)=0.

» By using method of synthetic division, $f(x)=2(x+1)^3+(x+1)^2-10(x+1)+4$. Let $g(x)=f(x-1)=2x^3+x^2-10x+4$. By Descartes' Rule, g(x)=0 has exactly one negative root, say, c. Thus g(c)=f(c-1)=0; hence c-1(<0) is a negative root of f(x)=0. Since there are 2 variations of signs in the sequence of coefficients of f(-x)and since c-1 is a negative root of f(x)=0, f(x)=0 has two negative roots. Also, f(x)=0 has exactly one positive root ,by Descartes' rule.

Sturm's Method of location of real roots of a polynomial equation with real coefficients

Let f be a polynomial with real coefficients and f_1 be its first derivative. Let the operation of finding the h.c.f. of f and $f_1$be performed with the following modifications: The sign of each remainder is to be changed before it is used as the next divisor and the sign of the last remainder is also to be changed. Let the modified remainders be denoted $f_2,\ldots,f_r$. $f,f_1,f_2,\ldots,f_r$ are called Sturm's functions. During the process of finding Sturm's functions, at any step, we can multiply by positive constant but not by a negative constant.

Sturm's Theorem

Theorem 1.11 Let f be a polynomial with real coefficients and a,b be real numbers, a<b. The number of real roots of f(x)=0 lying between a and b (a multiple root, if there be any, being counted only once) is equal to the excess of the number of changes of signs in the sequence of Sturm's functions $f,f_1,\ldots,f_r$ when x=a over the number of changes of signs in the sequence when x=b.

Example1.48 Find the number and position of the real roots of the equation $x^3-3x+1=0$.

Let $f(x)=x^3-3x+1$. $f^{(1)}(x)=3(x^2-1)$. $f_1(x)=x^2-1$.The remainder on dividing f by f_1 is $-2x+1$. Thus $f_2=2x-1$. Dividing $2f_1$ by f_2 (and multiplying by 2 at an intermediate step), the remainder is -3; hence f_3 is 1.

	f	f_1	f_2	f_3	No.of changes of sign
$-\infty$	-	+	-	+	*3*
0	+	-	-	+	2
∞	+	+	+	+	0

Hence the equation has 3 real roots,(3-2=)1 negative root and (2-0=)2 positive roots.

Location of roots

	f	f_1	f_2	f_3	No.of changes of sign
-2	-	+	-	+	3
-1	+	0	-	+	2(0 to be treated as continuation)

0	+	-	-	+	2
1	-	0	+	+	1
2	+	+	+	+	0

Thus one root lies between -2 and -1; one lies between 0 and 1 and one lies between 1 and 2.

NoteIf at any stage of finding out the sequence of Sturm's sequence, we obtain a function all of whose roots are complex, then the h.c.f. process need not be continued further and the determination and location of real roots will be possible from the set of functions $f, f_1, \ldots, f_s$. This is because $f_s$retains same sign for all values of x and no alteration in the number of changes of sign can take place in the sequence of functions beyond f_s.

Example1.49 Find the number and position of the real roots of the equation $x^4+4x^3-x^2-2x-5=0$.

$f(x)=x^4+4x^3-x^2-2x-5$, $f_1(x)=2x^3+6x^2-x-1$, $f_2(x)=7x^2+2x+9$. We can verify all the roots of $f_2=0$ are complex and leading coefficient of f_2 is positive; hence $f_2(x)>0$ for all real x. Hence the remaining Sturm functions need not be calculated.

	f	f_1	f_2	No. of changes of sign
$-\infty$	+	-	+	2
0	-	-	+	1
∞	+	+	+	0

The equation has two real roots, one positive and one negative.

Relations between roots and coefficients

Let $c_1, \ldots, c_n$ be the roots of the equation $a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n=0$. By factor theorem,

$a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n=a_0(x-c_1)(x-c_2)\ldots(x-c_n)$.

Equating coefficients of like powers of x,$a_1=a_0(-\sum c_1)$, $a_2=a_0\sum c_1\, c_2, \ldots, a_n=a_0\,(-1)^n c_1c_2\ldots c_n$. Hence

$\sum c_1=-\frac{a_1}{a_0}, \sum c_1\, c_2=\frac{a_2}{a_0}, \ldots, c_1c_2\ldots c_n=(-1)^n\frac{a_n}{a_0}$.

Example1.50 Solve the equation $2x^3-x^2-18x+9=0$ if two of the roots are equal in magnitude but opposite in signs.

» Let the roots be -a, a, b .Using relations between roots and coefficients, $b=(-a)+a+b=\frac{1}{2}$ and $-a^2b=-\frac{9}{2}$. Hence $a^2=9$, that is, $a=\pm3$. Hence the roots are $3,-3,\frac{1}{2}$.

Example1.51 Solve $x^3+6x^2+11x+6=0$ given that the roots are in A.P.

Symmetric functions of roots

A function f of two or more variables is symmetric if f remains unaltered by an interchange of any two of the variables of which f is a function. A symmetric function of the roots of a plolynomial equation which is sum of a number of terms of the same type is represented by any one of its terms with a sigma notation before it: for example, if a,b,c be the roots of a cubic polynomial, then $\sum a^2$ will stand for $a^2+b^2+c^2$.

Example1.52 If a,b,c be the roots of $x^3+px^2+qx+r=0$, find the value of (1)$\sum a^2$, (2)$\sum a^2 b$, (3) $\sum a^3$,(4)$\sum a^2 b^2$,(5) $\sum \frac{1}{a}$,(6) $\sum \frac{1}{ab}$,(7) $\sum \frac{1}{a^2}$.

» (1)$\sum a^2=(\sum a)^2 - 2\sum ab=p^2-2q$, (2) $\sum a^2 b=\sum a \sum a b$ $-3abc=-pq+3r$, (3)$\sum a^3=\sum a^2 \sum a-\sum a^2 b$, (4) $\sum a^2 b^2=(\sum ab)^2-2abc\sum a$, (5) $\sum \frac{1}{a}=\frac{\sum ab}{abc}$, (6) $\sum \frac{1}{ab}=\frac{\sum a}{abc}$, (7) $\sum \frac{1}{a^2}=\left(\sum \frac{1}{a}\right)^2 - 2\sum \frac{1}{ab}$.

Theorem 1.11(Newton) Let $a_1,\ldots,a_n$ be the roots of $x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0$, $s_r=a_1^r+\ldots+a_n^r$ where r is a non-negative integer. Then

(1) $s_r+p_1s_{r-1}+p_2s_{r-2}+\ldots+p_{r-1}s_1+rp_r=0$, if $1\le r \le n$
(2) $s_r+p_1s_{r-1}+p_2s_{r-2}+\ldots+p_ns_{r-n}=0$, if $r\ge n$

Example1.53 If a_1,a_2,a_3,a_4 be the roots of the equation $x^4+p_2x^2+p_3x+p_4=0$, find the value of (1)$\sum a_1^3$, (2)$\sum a_1^4$,(3)$\sum a_1^6$.

»(1)By Newton's Theorem, $s_3+p_2s_1+3p_3=0$. Here $s_1=0$.Thus $s_3=-3p_3$.

(2)By Newton's Theorem, $s_4+p_2s_2+p_3s_1+4p_4=0$. Here $s_1=0$ and $s_2=-2p_2$. Thus $s_4=2(p_2^2-2p_4)$

(3)$s_6+p_2s_4+p_3s_3+p_4s_2=0$. Hence $s_6=6p_4p_2+3p_3^2-2p_2^3$.

Transformation of equations

When a polynomial equation is given, it may be possible , without knowing the individual roots, to obtain a new equation whose roots are connected with those of the given equation by some assigned relation. The method of finding the new equation is said to be a transformation. Study of the transformed equation may throw some light on the nature of roots of the original equation.

(1) Let $c_1,\ldots,c_n$ be the roots of $a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n=0$; to obtain the equation whose roots are $mc_1,mc_2,\ldots,mc_n$. ($m=-1$ is an interesting case)
» Let $d_1=mc_1$. Since c_1 is a root of $a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n=0$, we have $a_0c_1^n+a_1c_1^{n-1}+\ldots+a_{n-1}c_1+a_n=0$. Replacing c_1 by d_1/m, we get $a_0d_1^n+ma_1d_1^{n-1}+m^2a_2d_1^{n-2}+\ldots+m^{n-1}a_{n-1}d_1+m^na_n=0$. Thus the required equation is $a_0x^n+ma_1x^{n-1}+\ldots+m^{n-1}a_{n-1}x+m^na_n=0$.

(2) Let $c_1,\ldots,c_n$ be the roots of $a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n=0$ and let $c_1c_2\ldots c_n\neq0$; to obtain the equation whose roots are $\frac{1}{c_1},\ldots,\frac{1}{c_n}$.
» Let $d_1=\frac{1}{c_1}$. So $c_1=\frac{1}{d_1}$. Substituting in $a_0c_1^n+a_1c_1^{n-1}+\ldots+a_{n-1}c_1+a_n=0$, we get $a_0+a_1d_1+a_2d_1^2+\ldots+a_{n-1}d_1^{n-1}+a_n\ d_1^n=0$. Thus $\frac{1}{c_1},\ldots,\frac{1}{c_n}$ are the roots of $a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0=0$.

(3) Find the equation whose roots are the roots of $f(x)=x^4-8x^2+8x+6=0$, each diminished by 2.
» $f(x)=(x-2)^4+8(x-2)^3+16(x-2)^2+8(x-2)+6=0$(by method of synthetic division). Undertaking the transformation $y=x-2$, the required equation is $y^4+8y^3+16y^2+8y+6=0$

Example1.54 If a,b,c be the roots of the equation $x^3+qx+r=0$, find the equation whose roots are (1) $a(b+c),b(c+a),c(a+b)$, (2) a^2+b^2,b^2+c^2,c^2+a^2, (3) $b+c-2a,c+a-2b,a+b-2c$.

(1) »$a(b+c)=\sum ab-bc=q-\frac{abc}{a}=q+\frac{r}{a}$. Thus the transformation is $y=q+\frac{r}{x}$. sustituting $x=\frac{r}{y-q}$ in $x^3+qx+r=0$ and simplifying, we obtain the required equation.

(2) »$a^2+b^2=\sum a^2-c^2=-2\sum ab-c^2=-2q-c^2$; hence the transformation is $y=-2q-x^2$, or, $x^2=-(y+2q)$. the given equation can be written as $x^2(x^2+q)^2=r^2$; thus the transformed equation is $(y+2q)(y+q)^2=-\ r^2$.

(3) »$b+c-2a=\sum a-3a=-3a$; the transformation is $y=-3x$.

Example1.55 Obtain the equation whose roots exceed the roots of $x^4+3x^2+8x+3=0$ by 1. Use Descartes' Rule of signs to both the equations to find the exact number of real and complex roots of the given equation.

» Let f(x)= $x^4+3x^2+8x+3=(x+1)^4-4(x+1)^3+9(x+1)^2-2(x+1)-1$ (by method of synthetic division). By Descartes' rule, f(-x) has two variations of signs in its coefficients and hence f(x)=0 has either two negative roots or no negative roots; also f(x)=0 has no positive root(since there is no variation of signs in the sequence of coefficients of f). Undertaking the transformation y=x+1, f(x)=0 transforms to $g(y)=y^4-4y^3+9y^2-2y-1$; considering g(-y), by Descrtes' Rule, g(y)=0 has a negeative root, say,a. Then f(x)=0 has a-1 as a negative root; the conclusion is f(x)=0 has two negative root, no positive root, does not have 0 as one of its roots and consequently exactly two complex conjugate roots (since coefficients of f are all real).

Example1.56 Find the equation whose roots are squares of the roots of the equation $x^4-x^3+2x^2-x+1=0$ and use Descartes'rule of signs to the resulting equation to deduce that the given equation has no real root.

»The given equation, after squaring, can be written as $(x^4+2x^2+1)^2=x^2(x^2+1)^2$; undertaking the transformation $y=x^2$, the transformed equation is $(y^2+2y+1)^2=y(y+1)^2$, that is, $y^4+3y^3+4y^2+3y+1=0$. The transformed equation, whose roots are squares of the roots of the original equation, has no nonnegative root by Descartes' Rule of signs; hence the original equation has no real root.

Example1.57 The roots of the equation $x^3+px^2+qx+r=0$ are a,b,c. Find the equation whose roots are a+b-2c,b+c-2a,c+a-2b. Deduce the condition that the roots of the given equation may be in A.P.

»a+b-2c=$\sum a-3c$=-p-3c. We undertake a transformation y=-p-3x, or,x=$-\frac{y+p}{3}$. Thus the equation whose roots are a+b-2c,b+c-2a,c+a-2b is $(y+p)^3-3p(y+p)^2+9q(y+p)-27r=0$. If the roots of given equation are in A.P., at least one root of transformed equation is zero, that is , the product of all the roots of the transformed equation is 0. Hence the condition is $2p^3-9pq+27r=0$.

Example1.58 Find the equation whose roots are cube of the roots of the equation $x^3+4x^2+1=0$.

»Let the roots of the given equation be a,b,c. Then $x^3+4x^2+1=(x-a)(x-b)(x-c)$. In this identity, we replace x by xw and xw^2, where w is the imaginary cube roots of unity, to obtain

$x^3+1+4x^2=(x-a)(x-b)(x-c)$

$x^3+1+4x^2w^2=(xw-a)(xw-b)(xw-c)=(x-aw^2)(x-bw^2)(x-cw^2)$

$x^3+1+4x^2w=(xw^2-a)(xw^2-b)(xw^2-c)=(x-aw)(x-bw)(x-cw)$

Multiplying respective sides,

$(x^3+1)^3+(4x^2)^3=(x^3-a^3)(x^3-b^3)(x^3-c^3)$.

Undertaking the transformation $y=x^3$, $(y+1)^3+4y^2=(y-a^3)(y-b^3)(y-c^3)$.

Thus the equation whose roots are a^3,b^3,c^3 is $(y+1)^3+4y^2=0$, or, $y^3+7y^2+3y+1=0$.

Cardan's Method of solving a cubic equation

Example1.59 Solve the equation:$x^3-15x^2-33x+847=0$.

Step 1 To transform the equation into one which lacks the second degree term.

Let $x=y+h$. The transformed equation is $y^3+(3h-15)y^2+(3h^2-30h-33)y+(h^3-15h^2-33h+847)=0$. Equating coefficient of y^2 to zero, $h=5$. Thus the transformed equation is $y^3-108y+432=0$ (*)

Step 2 Cardan's Method

Let $a=u+v$ be a solution of (*). Then $a^3-108a+432=0$ .also $a^3=u^3+v^3+3uv(u+v)=u^3+v^3+3uva$; so $a^3-3uva-(u^3+v^3)=0$. Comparing, $uv=36$ and $u^3+v^3=-432$. Hence u^3 and v^3 are the roots of $t^2+432t+36^3=0$. Hence $u^3=-216=v^3$. The three values of u are $-6,-6w$ and $-6w^2$, where w is an imaginarycube roots of unite. Since $uv=36$, the corresponding values of v are $-6,-6w^2,-6w$. Thus the roots of (*) are $-12,6,6$ and thus the roots of the given equation are (using $x=y+5$) $-7,11,11$.

Example1.60 Solve the equation:$x^3+6x^2+12x-9=0$.

Example1.61 Solve the equation:$x^3-3x-1=0$.

Example1.62 Solve the equation:$x^3-12x-65=0$.

FERRARI'S METHOD OF SOLVING A BIQUADRATIC EQUATION

We try to write down the given biquadratic equation $ax^4+4bx^3+6cx^2+4dx+e=0$ in the form $(ax^2+2bx+p)^2-(mx+n)^2=0$; equating coefficients of like powers of x, we get p,m,n; in the process we express the given biquadratic expression as product of two quadratic expressions and hence solve the given equation.

Example1.61 Solve the equation $x^4-10x^3+35x^2-50x+24=0$.

»We try to find p,m,n such that the given equation can be written in the form $(x^2-5x+p)^2-(mx+n)^2=0$. Equating coefficients of like powers of x, we get $35=25+2p-m^2$, $-50=-10p-2mn$ and $24=p^2-n^2$. Eliminating m and n,

$(p^2-24)(2p-10)=m^2n^2=(5p-25)^2$ (*)

p=5 is a solution of (*), hence $m=0, n=\pm 1$. Thus the given equation is $(x^2-5x+5)^2-1=0$, that is, $(x^2-5x+6)(x^2-5x+4)=0$. Hence the roots of given equation are 1,2,3,4.

Example1.62 Solve the equation $x^4+12x-5=0$.

»We try to find p,m,n such that the given equation can be written in the form $(x^2+p)^2-(mx+n)^2=0$; equating coefficients of like powers of x, we get $p^2-n^2=-5$, $2mn=-12$, $2p-m^2=0$. Eliminating m,n, we get $36=m^2n^2=(p^2+5)2p$; hence p=2. Thus $m=\pm 2$, $n=\pm 3$. Since $mn=-6<0$, we take m=2, n=-3. Thus the given equation can be written in the form $(x^2+2)^2-(2x-3)^2=0$, that is, $x^2+2x-5=0$ and $x^2-2x+5=0$. Hence the roots of given equation are $-1\pm\sqrt{2}$, $1\pm 2i$.

A B S T R A C T   A L G E B R A

CHAPTER III

SETS AND FUNCTIONS

In Mathematics, we define a mathematical concept in terms of more elementary concept(s).For example, the definition of perpendicularity between two straight lines is given in terms of the more basic concept of angle between two straight lines. The concept of set is such a basic one that it is difficult to define this concept in terms of more elementary concept. Accordingly, **we do not define 'set'** but to explain the concept intuitively we say: a set is a collection of objects having the property that given **any** abstract (the thought of getting 100%marks at the term-end examination) or concrete (student having a particular Roll No. of semester II mathematics general) objet, we can say

without any ambiguity whether that object belongs to the collection(collection of all thoughts that came to one's mind on a particular day or the collection of all students of this class) or not. For example, the collection of 'good' students of semester II will not be a set unless the criteria of 'goodness' is made explicit! The objects of which a set A is constituted of are called elements of the set A. If x is an element of a set A, we write **$x \in A$**; otherwise **$x \notin A$**. If every element of a set X is an element of set Y,X is a subset of Y, written as $X \subseteq Y$. X is a **proper** subset of Y if $X \subseteq Y$ and $Y \nsubseteq X$, written as $X \subsetneq Y$. For two sets X = Y iff (if and only if, bi-implication) $X \subseteq Y$ and $Y \subseteq X$. A set having no element is called null set, denoted by $\emptyset$.

Example1.1 $a \neq \{a\}$ (a letter inside an envelope is different from a letter without envelope!), $\{a\} \in \{a,\{a\}\}$, $\{a\} \subsetneq \{a,\{a\}\}$, $\emptyset \subset A$(the premise $x \in \emptyset$ of the implication $x \in \emptyset \Rightarrow x \in A$ is false and so the implication holds **vacuously** !), $A \subseteq A$, for every set A.

Set Operations: formation of new sets

Let X and Y be two sets. Union of X and Y, denoted by **$X \cup Y$**, is the set {a| $a \in X$ **or** $a \in Y$ **or** both}. Intersection of X and Y, denoted by **$X \cap Y$**, is the set {a| $a \in X$ **and** $a \in Y$}. The set difference of X and Y, denoted by **X-Y**, is the set {a| $a \in X$ **and** $a \notin Y$}. The set difference U-X is called complement of the set X, denoted by **X'**, where U is the universal set. The symmetric set difference of X and Y, denoted by **$X \Delta Y$**, is the set (X-Y) U(Y-X). For any set X, the power set of X, ***P*(X)**, is the set of all subsets of X. Two sets X and Y are disjoint iff $X \cap Y = \emptyset$. The Cartesian product of X and Y, denoted by **X X Y**, is defined as the set {(x,y)| $x \in X$, $y \in Y$} [(x,y) is called an ordered pair. Two ordered pairs (x,y) and (u,v) are equal, written (x,y) = (u,v), iff x = u and y = v]. If we take X = {1,2} and Y = {3}, then X X Y = {(1,3),(2,3)} $\neq$ {(3,1),(3,2)} = Y X X. Thus

Cartesian product between two **distinct** sets are **not necessarily** commutative (Is $\emptyset X \{1\} = \{1\} X \emptyset$ **?**).

Laws governing set operations

For sets X, Y, Z,

- **Idempotent laws**: $X \cup X = X$, $X \cap X = X$
- **Commutative laws**: $X \cup Y = Y \cup X$, $X \cap Y = Y \cap X$
- **Associative Laws**: $(X \cup Y) \cup Z = X \cup (Y \cup Z)$, $(X \cap Y) \cap Z = X \cap (Y \cap Z)$
- **Distributive laws**: $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$, $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$
- **Absorptive laws**: $X \cap (X \cup Y) = X$, $X \cup (X \cap Y) = X$
- **De' Morgan's laws**: $X-(Y \cup Z) = (X-Y) \cap (X-Z)$, $X-(Y \cap Z) = (X-Y) \cup (X-Z)$

Note: We may compare between usual addition and multiplication of real numbers on one hand and union and intersection of sets on the other. We see that the analogy is not complete e.g. union and intersection both are distributive over the other but addition is not distributive over multiplication though multiplication over addition is. Also $A \cup A = A$, for all set A but a.a = a does not hold for all real a.

Example1.2 Let A, B, C be three sets such that $A \cap C = B \cap C$ and $A \cap C' = B \cap C'$ holds. Prove that A = B.

» $A = A \cap U$ (U stands for the universal set conerned) $= A \cap (C \cup C')$(definition of complement of a set) $= (A \cap C) \cup (A \cap C')$(distributivity of $\cap$ over$\cup$) $= (B \cap C) \cup (B \cap C')$(given conditions)$= B \cap (C \cup C')$(distributivity of $\cap$ $over$ $\cup$)$=B$.

NOTE Make a habit of citing appropriate law at each step as far as practicable.

Example1.3 Let A, B, C be three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then prove B = C.

» $B = B \cup (A \cap B) = B \cup (A \cap C) = (B \cup A) \cap (B \cup C)$ (distributivity of ∪ over∩) $= (C \cup A) \cap (B \cup C) = C \cup (A \cap B) = C \cup (A \cap C) = C$.

Example1.4 $A\Delta C = B\Delta C$ implies A = B: prove or disprove.

NOTE: Proving will involve consideration of **arbitrary** sets A,B,C satisfying the given condition, whereas **disproving** consists of giving counter-examples of three **particular** sets A,B,C that satisfies the hypothesis $A\Delta C = B\Delta C$ but for which the conclusion A = B is false.

» This is a true statement. We first prove $A \subseteq B$. Let $x \in A$.

Case 1 $x \in C$. Then $x \notin (A-C) \cup (C-A) = A\Delta C = B\Delta C = (B-C) \cup (C-B)$. Thus $x \notin C - B$. Since $x \in C$, $x \in B$.

Case 2 $x \notin C$. $x \in (A-C) \subseteq A\Delta C = B\Delta C = (B-C) \cup (C-B)$. Since $x \notin C$, $x \notin C-B$. Thus $x \in B-C$. So $x \in B$.

Combining the two cases, we see $A \subseteq B$. Similarly, $B \subseteq A$. Combining, A = B.

Example1.5 Prove or disprove: $(A-B)' = (B-A)'$.

» This is a FALSE statement. **COUNTEREXAMPLE**: Let U = A = {1,2}, B = {1}.Then $(A-B)' = \{1\} \neq (B-A)' = \{1,2\}$.

Example1.6 Prove: $[(A-B) \cup (A \cap B)] \cap [(B-A) \cup (A \cup B)'] = \emptyset$

» By distributivity,$[(A-B) \cap (B-A)] \cup [(A-B) \cap (A \cup B)'] \cup [(A \cap B) \cap (B - A)] \cup [(A \cap B) \cap (A \cup B)]'] = \emptyset \cup [(A-B) \cap (A' \cap B')] \cup \emptyset \cup \emptyset = \emptyset$.

PRACTICE SUMS

1. Prove or disprove: $A \cup (B-C) = (A \cup B) - (A \cup C)$

2. Prove or disprove: A-C = B-C iff A∪C = B∪C.('IFF' stands for' if and only if')
3. Prove :A X (B∪C) = (A X B) ∪ (A X C)

NOTATION: N,Z,Q,R,C will denote set of all positive integers, integers, rational numbers ,real numbers and the complex numbers respectively.

BINARY RELATIONS

Definition1.1Abinary relationR from a set A to a set B is a subset of AxB.A binary relation (we shall often refer to as relation) from A to A is called a binary relation on A. If (a,b)∈R, we say a is R-related to b, written as aRb.

Example1.7 Let A={1,2,3} and R={(1,1),(1,3)}. Then 1R3 holds but 3R1 does not hold.

Example1.8 Let A be a set and $\mathcal{P}$(A) denote the power set of A. Given any two subsets X and Y of A, that is, X,Y∈$\mathcal{P}$(A), either X⊆Y or X⊈Y. Thus ⊆ is a binary relation on $\mathcal{P}$(A).

Example1.9 R={(x,y) ∈R^2/ $x^2+y^2=9$} is a relation on R.

Definition1.2Let R be a binary relation on a set A.

- R is reflexiveiffaRa holds ∀a∈A
- R is symmetriciffa,b∈A and aRbimplybRa
- R is transitiveiffa,b,c∈A, aRb,bRcimply aRc
- R is an equivalence relationon Aiff R is reflexive,symmetric and transitive.

Example1.10 Let R be a relation defined on Z by aRbiff ab≥0. R is reflexive, symmetric but not transitive: -5R0,0R7but -5R7 does not hold.

Example1.11 Let S be a binary relation on the set R of real numbers .

xSyiff	Reflexive	Symmetric	Transitive
y=2x	X	X	X
x<y	X	X	Yes
x≠y	X	Yes	X
xy>0	X	Yes	Yes
y≠ x +2	Yes	X	X
x≤y	Yes	X	Yes
xy≥0	Yes	Yes	X
x=y	Yes	Yes	Yes

Definition 1.2Let R be an equivalence relation on a set A. Let a∈A. [a]={x∈A/xRa} (⊆A) is the equivalence class determined by a with respect to R.

Definition 1.3Let A be a nonempty set and$\mathcal{P}$be a collection of nonempty subsets of A. Then $\mathcal{P}$ is a partition of Aiff

(1) for X,Y∈$\mathcal{P}$, either X=Y or X∩Y=∅ and (2) A=$\cup_{X\in \mathrm{P}} X$.

Theorem 1.1 :Let R be an equivalence relation on a set A. Then (1) [a]≠ ∅, ∀a∈A,

(2) b∈[a] iff [b]=[a],

(3) either [a]=[b]or [a] ∩ [b]=∅,

(4) A=$\bigcup_{a\in A}[a]$.

Thus {[a]/ a∈A} is a partition of A.

Proof: (1) since R is reflexive, (a,a)∈R ∀a∈A. Thus a∈[a]. Hence [a]≠ ∅, ∀a∈A.

(2) if [b]=[a], then b∈[b]=[a]. Conversely, let b∈[a]. Then aRb. For x∈[a], xRa holds and , by transitivity of R, xRb holds, that is , x∈[b]. Hence [a]⊆[b]. Similarly [b]⊆[a] can be proved . Hence [b]=[a].

(3) Let [a] ∩ [b]≠ ∅. Let x∈[a] ∩ [b]. Then aRx, xRb imply aRb, that is, [a]=[b].

(4) by definition, [a]⊆A, a∈A. Thus,$\bigcup_{a\in A}[a]$⊆A. conversely, let b∈A. Then b∈[b]⊆$\bigcup_{a\in A}[a]$. Thus A⊆$\bigcup_{a\in A}[a]$.Hence A=$\bigcup_{a\in A}[a]$.

Theorem 1.2 :Let $\mathcal{P}$ be a partition of a given set A. Define a relation R on A as follows:

for all a,b∈A, aRbiff there exists B∈$\mathcal{P}$ such that a,b∈B.

Then R is an equivalence relation on A.

Proof: Left as an exercise.

Example1.12 Verify whether the following relations on the set R of real numbers are equivalence relations: (1) aRbiff$|a-b|>0$, (2) aRb iff 1+ab>0, (3) aRb iff$|a|$≤b

Solution: (1)R is neither reflexive nor transitive but symmetric: 1R0 and 0R1 hold but 1R1 does not hold.

(2) R is reflexive and symmetric but not transitive:

3R($-\frac{1}{9}$) and ($-\frac{1}{9}$)$R(-6)$hold but 3R(-6) does not hold.

(3)(-2)R(-2)does not hold: not reflexive. -2R5 holds but 5R-2 does not: R not symmetric. Let pRq and qRs hold. Then $|p|$≤q≤$|q|$≤s imply pRs hold.

Example1.13 Verify whether the following relations on the set Z of integers are equivalence relations: (1) aRbiff$|a-b|\leq 3$, (2) aRb iff a-b is a multiple of 6, (3) aRbiff a^2-b^2 is a multiple of 7, (4) aRb iff $|a|=|b|$, (5) aRb iff 2a+b=41.

Example1.14 Let X≠ ∅. Prove that the following conditions are equivalent: (1) Ris an equivalence relation on X, (2) R is reflexive and for all x,y,z∈X, xRy and yRz imply zRx, (3)R is reflexive and for all x,y,z∈X, xRy and xRz imply yRz.

Example1.15 A relation R on the set of all nonzero complex numbers is defined by uRv iff $\frac{u-v}{u+v}$ is real. Prove that R is not an equivalence relation .

Solution: Note that iR2i, 2iR(-i) but (i,-i) does not belong to R.

Example1.15 A relation S on R^2 is defined by $(a_1,b_1)S(a_2,b_2)$ iff $\sqrt{a_1{}^2 + b_1{}^2} = \sqrt{a_2{}^2 + b_2{}^2}$. Prove that S is an equivalence relation and find equivalence class [(1,1)].

A **function** from a set A to a set B, denoted by f: A→B, is a correspondence between elements of A and B having the properties:

- ✓ For **every** x∈A, the corresponding element f(x) ∈ B. f(x) is called the image of x under the correspondence f and x is called a pre-image of f(x). A is called domain and B is called the **co-domain** of the correspondence. Note that we differentiate between f, the correspondence, and f(x), the image of x under f.
- ✓ For a **fixed** x∈A, f(x) ∈ B is unique. For two different elements x and y of A, images f(x) and f(y) may be same or may be different.

In brief, **a function is a correspondence under which**

- ➢ **both existence and uniqueness of image of all elements of the domain is guaranteed but**
- ➢ **neither the existence nor the uniqueness of pre-image of some element of co-domain is guaranteed**.

NOTATION Let f:A→B. For y∈B, $f^{-1}(\{y\}) = \emptyset$, if y has no preimage under f and stands for the set of all preimages if y has at least one preimage under f. For two elements y_1,y_2∈B, $f^{-1}(\{y_1,y_2\}) = f^{-1}(\{y_1\}) \cup f^{-1}(\{y_2\})$.

For C⊆A, $f(C) = \{f(c) | c \in C\}$, f(A) is called the range of f.

Example1.7 Prove that $f(A\cap B) \subseteq f(A) \cap f(B)$; give a counterexample to establish that the reverse inclusion **may not** hold.

» $y \in f(A\cap B) \Rightarrow y = f(x)$, $x \in A\cap B \Rightarrow$ $y = f(x)$, $x \in A$ and $x \in B \Rightarrow y \in f(A)$ and $y \in f(B) \Rightarrow y \in f(A) \cap f(B)$. Hence $f(A\cap B) \subseteq f(A) \cap f(B)$. Consider the counterexample: $f: R\rightarrow R$, $f(x) = x^2$, $A = \{2\}$, $B = \{-2\}$.

Example1.8 Let $f: R\rightarrow R$, $f(x) = 3x^2-5$. $f(x) = 70$ implies $x = \pm 5$. Thus $f^{-1}\{70\} = \{-5, 5\}$. Hence $f[f^{-1}\{70\}] = \{f(-5), f(-5)\} = \{70\}$. Also, $f^{-1}(\{-11\}) = \emptyset$ $[x\in f^{-1}(\{-11\})\Rightarrow 3x^2-5=-11\Rightarrow x^2=-2]$.

Example1.9 Let $g: R\rightarrow R$, $g(x) = \frac{x}{x^2+1}$. Find $g^{-1}(\{2\})$.

PRACTICE SUMS

Prove that (1) $f(A\cup B) = f(A) \cup f(B)$,

(2) $f^{-1}(B_1\cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$,

(3) $f^{-1}(B_1\cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

A function under which uniqueness of pre-image is guaranteed is called an **injective** function. A function under which existence of pre-image is guaranteed is called a **surjective** function. Put in a different language, $f: A\rightarrow B$ is injective iff $a_1, a_2\in A$, $f(a_1) = f(a_2)$ implies $a_1 = a_2$. The function f is surjective iff co-domain and the range coincide. A function which is both injective and surjective is called **bijective.**

NOTE: The injectivity, surjectivity and bijectivity depends very much on the domain and codomain sets and may well change with the variation of those sets even if expression of the function remains unaltered e.g. $f:Z\rightarrow Z$, $f(x) = x^2$ is not injective though $g:N\rightarrow Z$, $g(x) = x^2$ is injective.

Example1.10 f: R→R, f(x) = $x^2 - 3x+4$. $f(x_1) = f(x_2)$ implies $(x_1-x_2)(x_1+x_2-3) = 0$. Thus f(1) = f(2) though 1≠2; hence f is not injective[Note: for establishing non-injectivity, it is sufficient to consider particular values of x]. Let y∈R and x∈ $f^{-1}\{y\}$. Then y = f(x) = $x^2 - 3x+4$. We get a quadratic equation $x^2 - 3x+(4-y) = 0$ whose roots, considered as a quadratic in x, give preimage(s) of y. But the quadratic will have real roots if the discriminant 4y-7≥0, that is , only when y≥7/4. Thus, for example, $f^{-1}\{1\} =\emptyset$. Hence f is not surjective.

If f:A→B and g:B→C, we can define a function g_0f:A→C, called the **composition** of f and g, by $(g_0f)(a) = g(f(a))$, a∈A.

Example1.11 f:Z →Z and g: Z →Z by f(n) = $(-1)^n$ and g(n) = 2n. Then g_0f: Z →Z, $(g_0f)(n)=g((-1)^n) = 2(-1)^n$ and $(f_0g)(n) = (-1)^{2n}$. Thus $g_0f \neq f_0g$. Commutativity of composition of functions need not hold.

Note: Composition of functions, whenever is defined, is associative.

PRACTICE SUMS

Let f:A→B and g:B→ C.Then prove that:

(1) if f and g are both injective, then g_0f is so.
[hints $(g_0f)(x)=(g_0f)(y) \Rightarrow g(f(x))=g(f(y)) \Rightarrow f(x)=f(y) \Rightarrow x=y$]

(2) If g_0f is injective, then f is injective.

(3) [hints $f(x)=f(y) \Rightarrow g(f(x))=g(f(y)) \Rightarrow (g_0f)(x)=(g_0f)(y) \Rightarrow x=y$]

(4) if f and g are both surjective, then g_0f is so.
[hints $c\in C \Rightarrow \exists b\in B, c=g(b) \Rightarrow \exists a\in A, b=f(a) \Rightarrow c=g(b)=g(f(a))=(g_0f)(a)$]

(5) If g_0f is surjective, then f is surjective.

Verify whether following functions are surjective and / or injective:

(6) f:R→R, f(x) = $x|x|$

(7) f: (-1,1) →R, f(x) = $\frac{x}{1+|x|}$

Let f:A→B be a bijective function. We can define a function f^{-1}: B→ A by $f^{-1}(y)$ = x iff f(x) = y. Convince yourself that because of uniqueness and existence of preimage under f (since f is injective and surjective), f^{-1} is indeed a function. The function f^{-1} is called the **inverse function** to f. The graphs of f and f^{-1} for a given f can be seen here: ..\Documents\x8.mw

Note Graph of f^{-1} can be obtained by reflecting the graph of f about line x=y.

Example1.12 Let f: (0,1) →(1/2,2/3) be defined by f(x) = $\frac{x+1}{x+2}$. Verify that f is bijective (DO IT!).

[explanation:f(x)=f(y)⇒ $\frac{x+1}{x+2}=\frac{y+1}{y+2}$⇒x=y. Let c∈(1/2,2/3). If possible, let x be a pre-image of c under f, that is, f(x)=c. then c=$\frac{x+1}{x+2}$ implying x=$\frac{1-2c}{c-1}$ ∈(0,1) since $-1/2<c-1<-1/3$, $-1/3<1-2c<0$]. f^{-1}: (1/2,2/3) → (0,1) is to be found. Now, let $f^{-1}(y)$ = x, y∈ (1/2,2/3) . Then f(x) = y. So $\frac{x+1}{x+2}$=y and hence x = $\frac{1-2y}{y-1}$ = $f^{-1}(y)$.

BINARY OPERATIONS

Definition 1.10 Let A≠∅. A binary operation‘$_0$’ on A is a function from Ax A to A. In other words , a binary operation ‘$_0$’ on A is a rule of correspondence that assigns to each ordered pair (a_1,a_2)∈ A x A, some element of A, which we shall denote by $a_1\,{}_0a_2$. Note that $a_1\,{}_0a_2$need not bedistinct from a_1 or a_2.

Example1.26 Subtraction is a binary operation on Z but not on N; division is a binary operation on the set Q^* of all nonzero rational number but not on Z.

Definition 1.11 Let $_0$be a binary operation on A≠∅.

$(A,_0)$is called a mathematical system.

$_0$ is commutative iff $x_0y = y_0x$ holds ,for all x,y∈A.

$_0$ is associativeiff $x_0(y_0z) = (x_0y)_0z$ holds for all x,y,z∈A.

An element e∈A is aleftidentity of the system $(A,_0)$ iff $e_0x = x$ holds ∀x∈A.

An element e∈A is a right identity of the system $(A,_0)$ iff $x_0e = x$ holds ∀x∈A.

An element in a system which is both a left and a right identity of the system is called anidentityof the sytem.

$(A,_0)$ be a system with an identity e and let x, y∈A such that $x_0y = e$ holds. Then y(x) is called aright inverseto x(x is a left inverseof y respectively) in $(A,_0)$.y ∈ A is an inverse to x∈A iff $x_0y=y_0x=e$.

Example1.27 Consider the system $(R,_0)$ defined by $x_0y = x$, ∀x,y∈R (R stands for the set of real numbers). Verify that $_0$ is non-commutative, associative binary operation and that $(R,_0)$ has no left identity though$(R,_0)$ has infinite number of right identity.

Example1.28 Verify that subtraction is neither associative nor commutative binary operation on Z. (Z,-) does not have any identity.

Example1.29 Consider the system $(Z,*)$ where the binary operation $*$ is defined by $a*b = |a + b|$, a,b∈Z. Verify that $*$ is commutative but not associative [note: to show that $*$is not associative, it is sufficient to give an example, say, $\{(-1)*2\}*(-3) \neq (-1)*\{2*(-3)\}$].

$(Z,*)$ does not have an identity.

Example1.30 (R,+) is commutative, associative, possesses an identity element 0 and every element of (R,+) has an inverse in (R,+).

Note: From examples 1.12 to 1.15 it is clear that associativity and commutativity of a binary operation are properties independent of each other, that is, one can not be deduced from the other.

Example1.31 Let 2Z denote set of all even integers. 2Z, under usual multiplication, form a system which is associative, commutative but possesses no identity.

Example1.32 Let $M_2(Z)$ $=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in Z\right\}$. $M_2(Z)$ under usual matrix addition forms a system which is commutative, associative. $(M_2(Z),+)$ possesses an identity, namely the null matrix, and every element in $(M_2(Z),+)$ has an inverse in $(M_2(Z),+)$.

Example1.33 Let GL(2,R) denote the set of all 2x2 real non-singular matrices under usual matrix multiplication. The system is associative, non-commutative, possesses an identity and every element has an inverse in the system.