

**SEMESTER I MATHEMATICS GENERAL**

**CLASSICAL ALGEBRA**

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**CHAPTER I**

**COMPLEX NUMBERS: DE MOIVRE’S THEOREM**

**Definition 1.1** A complex number z is an ordered pair of real numbers (a, b): a is called Real part of z, denoted by Re z and b is called imaginary part of z, denoted by Im z. If Re z=0, then z is called purely imaginary ; if Im z =0, then z is called real. On the set C of all complex numbers, the relation of equality and the operations of addition and multiplication are defined as follows:

(a, b)=(c, d) iff a=c and b=d, (a, b)+(c, d)=(a+c, b+d), (a, b).(c, d)= (ac-bd, ad+ bc)

The set C of all complex numbers under the operations of addition and multiplication as defined above satisfies following properties:

* For z1,z2,z3C, (1) (z1+z2)+z3=z1+(z2+z3)(associativity), (2)z1+(0,0)=z1(existence of identity), (3) for z=(a, b)C, there exists –z=(-a,-b)C such that (-z)+z=z+(-z)=(0,0)(existence of inverse), (4)z1+z2=z2+z1(commutativity).
* For z1,z2,z3C, (1) (z1. z2).z3=z1.(z2.z3)(associativity), (2)z1.(1,0)=z1(existence of identity), (3) for z=(a, b)C,z(0,0), there exists C such that z.=.z=1(existence of inverse), (4)z1.z2=z2.z1(commutativity).
* For z1, z2, z3C, z1.(z2+z3)=(z1.z2)+(z1.z3)(distributivity).

**Few Observations**

1. Denoting the complex number (0,1) by i and identifying a real complex number (a,0) with the real number a, we see z=(a, b)=(a,0)+(0,b)=(a,0)+(0,1)(b,0) can be written as z=a+ib.
2. For two real numbers a, b , a2+b2=0 implies a=0=b; same conclusion need not follow for two complex numbers, for example, 12+i2=0 but 1≡(1,0)(0,0) ≡0 and i=(0,1)(0,0) (≡ denotes identification of a real complex number with the corresponding real number).
3. For two complex numbers z1,z2, z1z2=0 implies z1=0 or z2=0.
4. i2=(0,1)(0,1)=(-1,0) ≡-1.
5. Just as real numbers are represented as points on a line, complex numbers can be represented as points on a plane: z=(a, b)P: (a,b). The line containing points representing the real complex numbers (a,0), a real, is called the real axis and the line containing points representing purely imaginary complex numbers (0,b)=(0,1)(b,0)≡ib is called the imaginary axis. The plane on which the representation is made is called Gaussian Plane or Argand Plane.

**Definition 1.2** Let z=(a, b) ≡a+ ib. The conjugate of z, denoted by , is (a,-b) ≡a-ib.

Geometrically, the point (representing) is the reflection of the point (representing) z in the real axis. The conjugation operation satisfies the following properties:

1. =z , (2) , (3) , (4) , (4) z+=2 Rez, z-=2i Im(z)

**Definition 1.3** Let z=(a, b) ≡a+ib. The modulus of z, written as , is defined as .

Geometrically, represents the distance of the point representing z from the origin (representing complex number (0,0) ≡0+i0). More generally, represents the distance between the points z1 and z2. The modulus operation satisfies the following properties:

1. , (2) (3) (4)

### Geometrical representation of complex numbers: the Argand Plane

Let z=a+ib be a complex number. In the Argand plane, z is represented by the point whose Cartesian co-ordinates is (a, b) referred to two perpendicular lines as axes, the first co-ordinate axis is called the real axis and the second the imaginary axis. Taking the origin as the pole and the real axis as the initial line, let (r,) be the polar co-ordinates of the point (a,b). Then a=r cos, b=r sin. Also r==. Thus z=a+ib=(cos+isin): this is called **modulus-amplitude form** of z. For a given z0, there exist infinitely many values of differing from one another by an integral multiple of 2: the collection of all such values of for a given z0 is denoted by Arg z or Amp z. The **principal value** of Arg z , denoted by arg z or amp z, is defined to be the angle from the collection Arg z that satisfies the condition -. Thus Arg z={arg z+2n: n an integer}. arg z satisfies following properties: (1) arg(z1z2)=arg z1+arg z2+2k, where k is a suitable integer from the set {-1,0,1} such that - arg z1+arg z2+2k, (2) arg arg z1-arg z2+2k, where k is a suitable integer from the set{-1,0,1} such that - argz1-argz2+2k.

arg(i.i)=arg(i)+arg(i)-2.1., =arg(i.i)=arg(-i.-i)=arg(-i)+arg(-i)+2.1.

**Note** An argument of a complex number z=a+ ib is to be determined from the relations cos=a/, sin= b/ simultaneously and not from the single relation tan =b/a.

**Example 1.1** Find arg z where z=1+itan.

≻Let 1+itan=r(cos +i sin ). Then r2= sec2 . Thus r=- sec >0.So cos =- cos, sin =-sin . Hence =+. Since >, arg z=-2=-.

Geometrical representation of operations on complex numbers:

Addition Let P and Q represent the complex numbers z1=x1+iy1 and z2=x2+iy2 on the Argand Plane respectively. It can be shown that the fourth vertex R of the parallelogram OPRQ represents the sum z1+z2 of z1 and z2.

Product Let z1=) and z2=) where

–<,. Thus z1z2={cos(+)+isin(+)}. Hence the point representing z1z2 is obtained by rotating line segment OP ( where P represents z1) through arg z2 and then dilating the resulting line segment by a factor of . In particular , multiplying a complex number z by i=cos geometrically means rotating the directed line segment representing z by and, more generally, multiplying z by cos+isin means rotating directed line segment representing z by angle .

**Theorem1.1** (De Moivre’s Theorem) If n is an integer and is any real number, then (cos+i sin)n= cos n+i sin n. If n=, q natural, p integer, and q are realtively prime, is any real number, then (cos+i sin)n has q number of values, one of which is cos n+i sin n.

Proof **Case 1** Let n be a positive integer.

Result holds for n=1: (cos+i sin)1= cos 1+i sin 1. Assume result holds for some positive integer k: (cos+i sin)k= cos k+i sin k.Then (cos+i sin)k+1=(cos+i sin)k(cos+i sin)=( cos k+i sin k)( cos+i sin)= cos(k+1)+isin(k+1) Hence result holds by mathematical induction.

**Case 2** Let n be a negative integer, say, n= -m, m natural.

(cos+i sin)n=(cos+i sin)-m= (by case 1) = cos m-i sin m=cos(-m)+isin(-m)= cos n+i sin n.

**Case3** n=0: proof obvious.

**Case 4** Let n=, q natural, p integer, and q are realtively prime.

Let = cos +i sin. Then = (cos +i sin)q. Thus cos p+i sin p= cos q+i sin q. Thus q=2k+p, that is, =, k is any integer. Hence = cos()+isin(, where k=0,1,…,q-1 are the distinct q values (a value obtained by putting other integral values of k can be verified to be one of the values obtained by putting k=0,1,…,q-1).

**Some Applications of De’ Moivre’s Theorem**

1. Expansion of cos n, sin n and tan n where n is natural and is real.

cos n+i sin n=(cos+isin)n=cosn+ cosn-1isin+ cosn-2+…+insinn= (cosn- cosn-2+…)+i( cosn-1sin- cosn-3+…). Equating real and imaginary parts, cos n= cosn- cosn-2+… and sin n= cosn-1sin- cosn-3+…

1. Expansion of cosn and sinn in a series of multiples of where n is natural and .

Let x = cos . Then xn=cos n+isin n, x-n= cos n-isin n. Thus (2 cos)n=(x+n

=(xn+(xn-2+)+… =2 cos n+(2 cos(n-2))+…

Similarly, expansion of sinn in terms of multiple angle can be derived.

1. Finding n th roots of unity

To find z satisfying zn=1=cos (2k)+i sin(2k, where k is an integer. Thus z=[ cos(2k)+isin(2k]1/n=cos(, k=0,1,…,n-1; replacing k by any integer gives rise to a complex number in the set A={ cos(/ k=0,1,…,n-1}. Thus A is the set of all n th roots of unity.

**Example 1.2** Prove that 1 for all z satisfying

**»**We have ; hence 1 and -1; hence 1.

**Example 1.2** Solve x6+x5+x4+x3+x2+x+1=0

**»**We have the identity x6+x5+x4+x3+x2+x+1=. Roots of x7-1=0 are cos, k=0, 1,…,6. Putting k=0, we obtain root of x-1=0. Thus the roots of given equation are cos, k=1,…, 6.

**Example 1.3** Prove that the sum of 99 th powers of all the roots of x7-1=0 is zero.

**»**The roots of x7-1=0 are {1,,2,…,6}, where =cos+i sin. Thus sum of 99th powers of the roots is 1+99+(1+99+(==0, since =1 and 1.

**Example 1.4** If and arg z1+ argz2=0, then show that z1=.

**»**Let =r, arg z1=, then arg z2=-. Thus z1=r(cos +isin)=r(cos-isin)=.

**Example 1.5** For any complex number z, show that .

**»**Let z=x+iy.2(x2+y2)-(x+y)2=(x-y)20. Thus x2+y2 and so = , if x,y0. If x,y<0, then let x1=-x, y1=-y; then x1,y1>0. By above consideratitons, ==. Similarly other cases.

**Example 1.6** Prove that if the ratio is purely imaginary, then the point (representing )z lies on the circle whose centre is at the point and radius is .

**»**Let z=x+iy. Then . By given condition , =0, that is, .Thus z lies on the circle whose centre is at the point and radius is .

**Example 1.7** If the amplitude of the complex number is , show that z lies on a circle in the Argand plane.

**»**Let z=x+iy. Then =. By given condition, =1. On simplification, (x+1)2+(y-1)2=1. Hence z lies on the circle centred at (-1,1) and radius 1.

**Example 1.8** If z and z1 are two complex numbers such that z+z1 and zz1 are both real, show that either z and z1 are both real or z1=.

**»**Since zz1 is real, arg(zz1)=0; thus arg z =-arg z1=(say). Thus z=r(cos +i sinz1=r1(cos-i sin). Hence z+z1=(r+r1)cos+i(r-r1)sin. Since z+z1 is real, either sin=0 in which case z and z1 are both real or r=r1 in which case z1=.

**Example 1.9** If , prove that arg z1 and argz2 differ by or .

Thus (z1+z2)(= (z1-z2)( Or, **(1)**. Let z1=r1 (cos), z2=r2(cos). From (1), cos(-)=0 proving the result.

**Example 1.10** If A,B,C represent complex numbers z1,z2,z3 in the Argand plane and z1+z2+z3=0 and =, prove that ABC is an equilateral triangle.

**»**z1+z2=-z3. Hence , that is, ++2z1.z2=. By given condition, 2cos =, where is the angle between z1 and z2(considering z1,z2 as vectors).Thus cos =-, that is, =1200. Hence the corresponding angle of the triangle ABC is 600. Similarly other angles are 600.

**Example 1.11**  If (x,y) represents a point lying on the line 3x+4y+5=0, find the minimum value of .

**»** Find perpendicular distance of origin from the given straight line.

**Example 1.12** Let z and z1 be two complex numbers satisfying z= and =1. Prove that z lies on the imaginary axis.

**»** z1=. By given condition, . If z=x+iy, (x-1)2+y2=(x+1)2+y2; hence x=0. Thus.

**Example 1.13**  If z1, z2 are conjugates and z3,z4 are conjugates, prove that arg arg.

**»** Since z1, z2 are conjugates, arg z1+arg z2=0. Since z3, z4 are conjugates, arg z3+arg z4=0. Thus arg z1+arg z2=0= arg z3+arg z4. Hence arg z1-arg z4=arg z3-arg z2; thus result holds.

**Example 1.14**  complex numbers z1,z2,z3 satisfy the relation z12+z22+z32-z1z2-z2z3-z3z1=0 iff .

**»** 0=z12+z22+z32-z1z2-z2z3-z3z1=(z1+wz2+w2z3)(z1+w2z2+wz3), where w stands for an imaginary cube roots of unity. If z1+wz2+w2z3=0, then (z1-z2)=-w2(z3-z2); hence ==similarly other part.

Conversely, if , then z1,z2,z3 represent vertices of an equilateral triangle. Then z2-z1=(z3-z1)(cos 600+isin600), z1-z2=(z3-z2)( cos 600+isin600 ); by dividing respective sides, we get the result.

**Example 1.15** Prove that ), for two complex numbers z1,z2.

=+2z1; similarly =-2 z1; Adding we get the result.

**Example 1.16**  If cos+cos +cos =sin+sin+sin=0, then prove that (1) cos 3+cos3+cos 3=3cos(+), (2) ==3/2.

Let x=cos +i sin, y= cos +i sin, z= cos +i sin. Then x+y+z=0. Thus x3+y3+z3=3xyz. By De’ Moivre’s Theorem, (cos 3+ cos 3+cos 3)+i(sin 3+ sin 3+sin 3)=3[ cos()+isin(. Equating, we get result.

Let x=cos +i sin, y= cos +i sin, z= cos +i sin. Then x+y+z=0. Also hence xy+yz+zx=0. Thus x2+y2+z2=(x+y+z)2-2(xy+yz+zx)= 0. By De’ Moivre’s Theorem, cos 2+cos2+cos2=0. Hence=3/2. Using sin2=1- cos2, we get other part.

**Example 1.17**  Find the roots of zn=(z+1)n, where n is a positive integer, and show that the points which represent them in the Argand plane are collinear.

Let w=Now zn=(z+1)n implies wn=1.Thus, w=cos,k=0,…,n-1.

So z= , k=1,…,n-1

=. Thus all points z satisfying zn=(z+1)n lie on the line x=-.

**Functions of a complex variable**

Exponential function

**Definition 1.4** For a complex number z=x+iy, exp(z)=ex(cos y +i sin y).

**Note**

1. If z=x+i0 is purely real, exp(z)=ex.
2. If z = 0+iy is purely imaginary, exp(z)=exp(iy)=cos y+i sin y.
3. Since ex, thus exp(z) is a non-zero complex number for every complex number z.  as x=Re z. Also =ex>x.
4. For every non-zero complex number u+iv=r(cos+isin), exp(ln r+i)=

eln r(cos+isin)=r(cos+isin)=u+iv. Thus exp:CC-{0+i0}.

1. exp(z1).exp(z2)=exp(z1+z2)

**»** Let z1=x1+iy1, z2=x2+iy2. exp(z1)exp(z2) =(cosy1+isiny1)(cosy2+isiny2) =[cos(y1+y2)+isin(y1+y2)]= exp(z1+z2).

1. exp(z1-z2)= ; in particular exp(-z)=.
2. If n be an integer, (exp z)n=exp(nz) (follows from property(5) and (6))
3. If n be an integer, exp(z+2ni)=exp(z).(follows from (5), since exp(2ni)=1)
4. Unlike the case of ex, which is always positive, exp(z) can be negative real number (as seen below).

**Example 1.18** Find all complex number z such that exp(z)=-1

**»** Let z=x+iy. Then ex(cos y +i siny)=-1. Thus ex cos y=-1, ex sin y=0. Squaring, adding gives e2x=1 whence ex=1(since ex>0); thus x=0. cos y=-1 and sin y=0 gives y=(2n+1). Thus z=(2n+1)i, n integer.

Logarithmic function

Let z be a non-zero complex number. Then there exists a complex number w such that exp (w)=z; further if exp(w)=z, then exp(w+2n)=z.

**Definition 1.5** For a given non-zero complex number z, we define Log z={wC/ exp(z)=w} and call the set Logarithm of z.

Let z=r(cos +isin),-<. Let w=u+ iv be a logarithm of z(that is, a member of Log z). Then exp(u+iv)= r(cos +isin). Thus eucos v=r cos, eusin v=r sin. Hence eu=r (by squaring and adding last two relations), cos v=cos, sinv=sin. So u=ln r, v=+2n. Thus Logz=ln+i(arg z+2n).

**Note**

1. Putting n=0 in the expression for Log z, we obtain **principal value** of Log z, denoted by log z. Thus log z= ln+iarg z.
2. exp(Log z)=z for z0; one of the values of Log(expz) is z, the other values are z+2ni,n integer.
3. For two distinct non-zero complex numbers z1,z2, Log(z1z2)=Log(z1)+Log(z2); log(z1z2)log(z1)+log(z2): take z1=i,z2=-1.

**»** Let z1=r1(cos+isin), z2= r2(cos+isin). Then Log z1=lnr1+i(+2n), Log z2=lnr2+i(+2m),Log(z1z2)= z1=ln(r1r2)+i(+2k) where n, m, k are integers. Log(z1)+Log(z2)=ln(r1r2)+i(++2q, where q=m+n. Since p,q are arbitrary integers, Log(z1z2)=Log(z1)+Log(z2) holds.

1. For two distinct non-zero complex numbers z1,z2, LogLog(z1)-Log(z2). loglog(z1)-log(z2): take z1=-1,z2=-i.

**»** Proof similar.

1. If z0 and m be a positive integer, Log zmmLogz: take z=i,m=2.

**Example 1.19** Verify Log(-i)1/2=1/2 Log(-i)

**»** -i=cos+isin. Log(-i)=[(2n]=( n)I, where n is an integer.

Two values of (-i)1/2 are +isin and +isin. Now Log[+isin]=(2m-)i and Log[+isin]= (2p)i=[(2p+1)]I, where m,p are integers.Thus the values of Log(-i)1/2 can be expressed as (n- )I, where n is an integer. Hence the result.

**Example 1.20** Prove that sin[ilog=.

**»**Let a+ib=r(cos+isin),-. Then a=r cos, b=r sin. Thus z=cos(-2)+isin(-2). Thus arg z=-2+2k, where k is an integer such that -<-2+2k. So log z=(-2+2k)i. hence sin(ilog z)=sin(2-2k)=sin2=.

Exponent function

**Definition 1.6** If w be a **nonzero** complex number and z be any complex number, then wz= exp(z Log w) . Since Log is many-valued, exponent is many-valued; principal value of wz is defined as exp(z log w).

**Note** If z1,z2,w are complex numbers, w0, then but p.v.=(p.v.)(p.v.).

**Example 1.21** Find values of ii.

**»** ii=exp[iLogi]=exp[i(2ni]=, n any integer.

**Example 1.22** If a, b are the imaginary cube roots of unity, prove that (1) p.v. of aa+p.v. of bb=, (2) p.v. of ab+p.v. of ba =.

**»**p.v. of aa=exp[a log a]=exp[a{ln1+i]], if a=-

=exp[-]=[cos]. Similarly, p.v. of bb=[cos].Hence (1) is proved. Similarly (2) can be proved.

**Example 1.23**  If iz=i, show that z is real and the general values of z are given by z=, m,n are integers.

**Example 1.24** Find all the values of and show that the values lie on a circle in the Argand plane.

Trigonometric function

**Definition1.7** From definition of exponential function, for real y, exp(iy)=cos y+ isin y, exp(-iy)=cosy-isiny. Thus for real y, cos y=, sin y=. Backed up by these results, for complex z, we define

cos z=, sin z=, tan z = and so on. We can prove the following results:

1. cos2z+sin2z=1
2. sin(z1+z2)=sinz1cosz2+cosz1sinz2, cos(z1+z2)=cosz1cosz2-sinz1sinz2.
3. sin(z+2)=sin z, sin(z+=-sin z
4. If x,y are real, sin(x+iy)=sin x cosh y+icos xsinh y, cos(x+iy)=cosx cosh y-isin xsinh y, where cosh x= and sinh x=. Thus 2=sin2x+sinh2y and 2=cos2x+sinh2y: since sinh y is an unbounded function, sin z and cos z are unbounded in absolute value. But if x is real, sin x and cos x are bounded functions.
5. cos(iz)=cosh z, sin(iz)=isinh z, cosh(iz)=cos z, sinh(iz)=i sin z, where cosh z= and sinh z=.
6. Unlike sin x, cos x(x real), sin z and cos z are unbounded.

**Example 1.25** Find all values of z such that cos z=0.

**»**Let z=x+iy. cos z=0 implies cos x cosh y=0, sin x sinh y=0. Since cosh y0, cos x=0. Thus x=(2n+1), n integer. Thus sin x=sin[(2n+1)]0. Hence sinh y=0. Thus y=0. Hence z=(2n+1), n integer.

**Example 1.26**  If x=log tan(, where is real, prove that =-iLog tan.

**»**Since is real, x real. ex=tan(=. Therefore, =. Thus =i, where t=exp. Hence ===tan. Thus t2=. So exp(i)=tan. =-iLog tan.

**Example 1.26**  If sin(x+iy)=u+iv (x,y,u,v are real), prove that u2cosec2x-v2sec2x=1, u2sech2y+v2cosech2y=1.

**»** sinx cos(iy)+cos x sin(iy)=u+iv. T hus u=sin x cosh y, v=cos x sinh y (since sin(iy)=i sinh y, cos(iy)=cosh y). Hence = cosh y, =sinh y. Since cosh2y- sinh2y=1, we get the first result. Again =sin x, =cos x. from cos2x+sin2x = 1, we get the second result.

**Example 1.26**  If x,y are real and =1, then prove that sin2x=sinh2y.

**»**cos(x+iy)=cos xcos(iy)-sinx sin(iy)=cos x cosh y-i sin x sinh y. Thus 1= =cos2x cosh2y+sin2x sinh2y=(1-sin2x)(1+sinh2y)+sin2x sinh2y=1-sin2x+sinh2y.

Inverse Trigonometric functions

**Definition1.8** Let z be a given complex number and w be a complex number such that sin w=z. Then cos w=. Thus exp(iw)=iz or, w=-iLog(iz). Since Log is a multi-valued function, Sin-1z=w=-iLog(iz) is a multiple valued function of z. The principal value of Sin-1z is obtained by choosing cos w=+ and by taking the principal value of the logarithm and is denoted by sin-1z: sin-1z=-I log(iz+).

**Example 1.27** Find Cos-1(2), cos-1(2).

**»**Let Cos-1(2)=z. Then cos z=2. sin z=i. Thus exp(iz)=cos z+i sin z=2=. When exp(iz)=2+, z=-i Log(2+)=-i[log(2++2ni]=2n-i log(2+). When exp(iz)=2-, z=-iLog(2-)=-i[log(2-+2ni]=2n-i log(2-) =2n+i log(2+. Thus Cos-1(2)= 2ni log(2+. Hence cos-1(2)=i log(2+).

**Example 1.28** If tan-1(x+iy)=a+ib, where x, y, a, b are real and (x,y)(0,1), prove that (1) x2+y2+2xcot 2a=1, (2) x2+y2+1-2ycoth 2b=0.

**»**tan (a+ib)=x+iy; thus tan(a-ib)=x-iy. Thus tan 2a=tan[(a+ib)+(a-ib)]=. Hence the first part. Again tan(2ib)=tan[(a+ib)-(a-ib)]=. But tan(2ib)==i tanh 2b. Hence the second part.­

**Example 1.28** If y=log tan(, then prove that x=-iLog tan(.

**»** Given ey= tan()=. By componendo and dividendo, =tan. Thus =tan, that is, tanh =tan . Hence =. Thus =. Using componendo and dividendo, exp(ix)=. Hence x=

-iLogtan(.

**Example 1.28** If Cos-1(x+iy)=u+iv, then prove that cos2u and cosh2v are the roots of the equation t2-t(1+x2+y2)+x2=0.

**»** x+iy=cos(u+iv)=cos u cosh v-i sin u sinh v. Thus x=cos u cosh v, y=-sin u sinh v. Hence 1+x2+y2=1+cos2u(1+sinh2v)+(1-cos2u)sinh2v=cos2u+cosh2v. hence the result.

**CHAPTER II**

**THEORY OF EQUATIONS**

An expression of the form a0xn+a1xn-1+…+an-1x+an, where a0,a1,…,an are real or complex constants, n is a nonnegative integer and x is a variable (over real or complex numbers) is a **polynomial in x**. If a00, the polynomial is of **degree** n

and a0xn is the **leading term** of the polynomial. A non-zero constant a0 is a polynomial of degree 0 while a polynomial in which the coefficients of each term is zero is said to be a zero polynomial and no degree is assigned to a zero polynomial.

**Equality** two polynomials a0xn+a1xn-1+…+an-1x+an and b0xn+b1xn-1+…+bn-1x+bn are equal iff a0=b0,a1=b1,…,an=bn.

**Addition** Let f(x)= a0xn+a1xn-1+…+an-1x+an, g(x)= b0xn+b1xn-1+…+bn-1x+bn. the sum of the polynomials f(x) and g(x) is given by f(x)+g(x)= a0xn+…+an-m-1xm+1+(an-m+b0)xm+…+(an+bm), if m<n

= (a0+b0)xn+…+(an+bn), if m=n

= b0xm+…+bm-n-1xn+1+(bm-n+a0)xn+…+(bm+an), if m>n.

**Multiplication** Let f(x)= a0xn+a1xn-1+…+an-1x+an, g(x)= b0xn+b1xn-1+…+bn-1x+bn. the product of the polynomials f(x) and g(x) is given by f(x)g(x)=c0xm+n+c1xm+n-1+…+cm+n, where ci=a0bi+a1bi-1+…+aib0. c0=a0b00; hence degree of f(x)g(x) is m+n.

**Division Algorithm** Let f(x) and g(x) be two polynomials of degree n and m respectively and nm. Then there exist two uniquely determined polynomials q(x) and r(x) satisfying f(x)=g(x)q(x)+r(x), where the degree of q(x) is n-m and r(x) is either a zero polynomial or the degree of r(x) is less than m. In particular, if degree of g(x) is 1, then r(x) is a constant, identically zero or non-zero.

**Remainder Theorem** If a polynomial f(x) is divided by x-a, then the remainder is f(a).

**»**Let q(x) be the quotient and r (constant)be the remainder when f is divided by x-a. Then f(x)=(x-a)q(x)+r is an identity. Thus f(a)=r.

**Factor Theorem** If f is a polynomial, then x-a is a factor of f(x) iff f(a)=0.

**»**By Remainder theorem, f(a) is the remainder when f is divided by x-a; hence, if f(a)=0, then x-a is a factor of f(x). Conversely, if x-a is a factor of f, then f(x)=(x-a)g(x) and hence f(a)=0.

**Example 2.1** Find the remainder when f(x)=4x5+3x3+6x2+5 is divided by 2x+1.

**»**The remainder on dividing f(x) by x-(-=x+ is f(-)=6. If q(x) be the quotient, then f(x)=q(x)(x+)+6=(2x+1)+6. Hence 6 is the remainder when f is divided by 2x+1.

Synthetic division

Synthetic division is a method of obtaining the quotient and remainder when a polynomial is divided by a first degree polynomial or by a finite product of first degree polynomials. Let q(x)=b0xn-1+b1xn-2+…+bn-1 be the quotient and R be the remainder when f(x)=a0xn+a1xn-1+…+an is divided by x-c. Then

a0xn+a1xn-1+…+an=( b0xn-1+b1xn-2+…+bn-1)(x-c)+R.

Above is an identity; equating coefficients of like powers of x, a0=b0, a1=b1-cb0, a2=b2-cb1,…,an-1=bn-1-cbn-2,an=R-cbn-1. Hence b0=a0,b1=a1+cb0,b2=a2+cb1,…,bn-1=an-1+cbn-2,R=an+cbn-1.

The calculation of b0,b1,…,bn-1,R can be performed as follows:

a0 a1 a2 … an-1 an

cb0 cb1 … cbn-2 cbn-1

b0 b1 b2 …. bn-1 R

**Example 2.2** Find the quotient and remainder when 2x3-x2+1 is divided by 2x+1.

- 2 -1 0 1

-1 1 -½

……………………………..

2 -2 1 ½

Thus 2x3-x2+1=(x+)(2x2-2x+1)+=(2x+1)(x2-x+)+; hence the quotient is x2-x+ and the remainder is .

Applications of the method

1. To express a polynomial f(x)=a0xn+a1xn-1+…+an as a polynomial in x-c.

Let f(x)=A0(x-c)n+A1(x-c)n-1+…+An. Then f(x)=(x-c)[A0(x-c)n-1+A1(x-c)n-2+…+An-1]+An. Thus on dividing f(x) by x-c, the remainder is An and the remainder is q(x)= A0(x-c)n-1+A1(x-c)n-2+…+An-1. Similarly if q(x) is divided by x-c, the remainder is An-1. Repeating the process n times, the successive remainders give the unknowns An,…,A1 and A0=a0.

1. Let f(x) be a polynomial in x. to express f(x+c) as a polynomial in x.

Let f(x)=a0xn+a1xn-1+…+an=A0(x-c)n+A1(x-c)n-1+…+An; by method explained above , the unknown coefficients can be found out in terms of the known coefficients a0,…,an. Now f(x+c)=A0xn+…+An.

**Example 2.3** Express f(x)=x3-6x2+12x-16=0 as a polynomial in x-2 and hence solve the equation f(x)=0.

**»** Using method of synthetic division repeatedly, we have

2 1 -6 12 -16 Hence f(x)=(x-2)3-8=0.

2 -8 8 Thus x-2=2,2w,2w2 (w is an imaginary cube roots of unity)

---------------------------- Hence x=4,2+2w,2+2w2.

1 -4 4 -8

2 -4

-----------------------

1 -2 0

2

------------

1. 0

**Example 2.4** If f(x) is a polynomial of degree 2 in x and a,b are unequal, show that the remainder on dividing f(x) by (x-a)(x-b) is.

**»** By division algorithm, let f(x)=(x-a)(x-b)q(x)+rx+s, where rx+s is the remainder. Replacing x by a and by b in turn in this identity, f(a)=ra+s, f(b)=rb+s; solving for r,s and substituting in the expression rx+s,we get required expression for remainder.

**Example 2.5** If x2+px+1 be a factor of ax3+bx+c, prove that a2-c2=ab. Show that in this case x2+px+1 is also a factor of cx3+bx2+a.

**»**Let ax3+bx+c=( x2+px+1)(ax+d) (\*) (taking into account the coefficient of x3). Comparing coefficients, d+ap=0,a+pd=b,d=c whence the result a2-c2=ab follows. Replacing x by 1/x in the identity (\*), we get cx3+bx2+a=( x2+px+1)(dx+a): this proves the second part.

If f(x) is a polynomial of degree n, then f(x)=0 is called a polynomial equation of degree n. If b is a real or complex number such that f(b)=0, then b is a **root** of the polynomial equation f(x)=0 or is a **zero** of the polynomial f(x). If (x-b)r is a factor of f(x) but (x-b)r+1 is not a factor of f(x), then b is a root of f(x)=0 of **multiplicity** r: a root of multiplicity 1 is called a simple root. Thus 2 is a simple root of x3-8=0 but 2 is a root of multiplicity 3 of (x-2)3(x+3)=0.

**Fundamental Theorem of Classical Algebra**

Every polynomial equation of degree1 has a root, real or complex.

**Corollary** A polynomial equation of degree n has exactly n roots, multiplicity of each root being taken into account.

**Corollary** If a polynomial f(x) of degree n vanishes for more than n distinct values of x, then f(x) =0 for all values of x(that is, f(x)=0 is an identity).

**Example 2.6** x2-4=(x+2)(x-2) is an identity since it is satisfied by more then two distinct values of x; in contrast (x-1)(x-2)=0 is an equality and not an identity.

**Theorem 2.1** If b is a multiple root of the polynomial equation f(x)=0 of multiplicity r, then b is a multiple root of f(1)(x)=0 of multiplicity r-1. Thus to find the multiple roots of a polynomial equation f(x)=0, we find the h.c.f. g(x) of the polynomials f(x) and f(1)(x). The roots of g(x)=0 are the multiple roots of f(x)=0, if there be any.

**Polynomial equations with Real Coefficients**

**Theorem 2.2** If a+ib is a root of multiplicity r of the polynomial equation f(x)=0 with real coefficients, then a-ib is a root of multiplicity r of f(x)=0.

**Note**1+i is a root of x2-(1+i)x=0 but not so is 1-i.

**Example 2.7** Prove that the roots of are all real.

**»** The given equation is ++=-5 (\*). Let a+ib be a root of the polynomial equation (\*) with real coefficients. Then a-ib is also a root of (\*).Thus ++=-5 and ++=-5. Subtracting,

-2ib[]=0 which gives b=0. Hence all roots of given equation must be real.

**Example 2.8** Solve the equation f(x)=x4+x2-2x+6=0 , given that 1+i is a root.

**»** Since f(x)=0 is a polynomial equation with real coefficients, 1-i is also a root of f(x)=0. By factor theorem,(x-1-i)(x-1+i)=x2-2x+2 is a factor of f(x). By division, f(x)=( x2-2x+2)(x2+2x+3). Roots of x2+2x+3=0 are -1i. Hence the roots of f(x)=0 are 1i, -1i.

**Theorem 2.3** If a+ is a root of multiplicity r of the polynomial equation f(x)=0 with rational coefficients, then a- is a root of multiplicity r of f(x)=0 where a,b are rational and b is not a perfect square of a rational number.

Since every polynomial with real coefficients is a continuous function from R to R, we have

**Theorem 2.4** (Intermediate Value Property) Let f(x) be a polynomial with real coefficients and a,b are distinct real numbers such that f(a) and f(b) are of opposite signs. Then f(x)=0 has an odd number of roots between a and b. If f(a) and f(b) are of same sign, then there is an even number of roots of f(x)=0 between a and b (even number includes zero).

**Example 2.9** Show that for all real values of a, the equation (x+3)(x+1)(x-2)(x-4)+a(x+2)(x-1)(x-3)=0 has all its roots real and simple.

**»**Let f(x)= (x+3)(x+1)(x-2)(x-4)+a(x+2)(x-1)(x-3). Then =, f(-2)<0, f(1)>0, f(3)<0,=. Thus each of the intervals (,-2),(-2,1),(1,3),(3,) contains a real root of f(x)=0. Since the equation is of degree 4, all its roots are real and simple.

**Theorem2.5** (**Rolle’s Theorem**) Let f(x) be a polynomial with real coefficients . Between any two distinct real roots of f(x)=0 ,there is at least one real root of f(1)(x)=0.

**Note**

1. Between two **consecutive** real roots of f(1)(x)=0, there is at most one real root of f(x)=0.
2. If all the roots of f(x)=0 be real and distinct, then all the roots of f(1)(x)=0 are also real and distinct.

**Example 2.10** Show that the equation f(x)=(x-a)3+(x-b)3+(x-c)3+(x-d)3=0, where a, b, c, d are real and not all equal , has only one real root.

**»** Since f(x)=0 is a cubic polynomial equation with real coefficients, f(x)=0 has either one or three real roots. If be a real multiple root of f(x)=0 with multiplicity 3, then is also a real root of f(1)(x)=3[(x-a)2+(x-b)2+(x-c)2+(x-d)2]=0, and hence =a=b=c=d (since ,a,b,c,d are real), contradiction. If f(x)=0 has two distinct real roots, then in between should lie a real root of f(1)(x)=0, contradiction since not all of a,b,c,d are equal. Hence f(x)=0 has only one real root.

**Example 2.11** Find the range of values of k for which the equation f(x)=x4+4x3-2x2-12x+k=0 has four real and unequal roots.

**»** Roots of f(1)(x)=0 are -3,-1,1. Since all the roots of f(x)=0 are to be real and distinct, they will be separated by the roots of f(1)(x)=0.Thus f(x)=0 has a root, and hence exactly one root, in each of the subintervals ( Now =,f(-3)=-9+k,f(-1)=7+k,f(1)=-9+k,=. Since f(-3)<0, f(-1)>0 and f(1)<0, -7<k<9.

**Example 2.12** If c1,c2,…,cn be the roots of xn+nax+b=0, prove that (c1-c2)(c1-c3)…(c1-cn)=n(c1n-1+a).

**»** By factor theorem, xn+nax+b=(x-c1)(x-c2)…(x-cn).Differentiating w.r.t. x, n(xn-1+a)= (x-c2)…(x-cn)+(x-c1)(x-c3)…(x-cn)+…+ (x-c2)(x-c3)…(x-cn).Replacing x by c1 in this identity, we obtain the result.

**Example 2.13** If a is a double root of f(x)=xn+p1xn-1+…+pn=0, prove that a is also a root of p1xn-1+2p2xn-2+…+npn=0.

**»** Since a is a double root of f(x)=0, both f(a)=0 and f(1)(a)=0 hold. Thus an+p1an-1+…+pn=0 (1) and nan-1+(n-1)p1an-2+…+pn-1=0(2). Multiplying both side of (1) by n and both side of (2) by a and subtracting, we get p1an-1+2p2an-2+…+npn=0.Hence the result.

**Example 2.14** Prove that the equation f(x)=1+x++…+=0 cannot have a multiple root.

**»** If a is a multiple root of f(x)=0, then 1+a++…+=0 and 1+a++…+=0;it thus follows that =0, so that a=0; but 0 is not a root of given equation. Hence no multiple root.

**Descartes’ Rule of signs for polynomials with real coefficients**

**Theorem 2.6** The number of positive roots of an equation f(x)=0 with real coefficients does not exceed the number of variations of signs in the sequence of the coefficients of f(x) and if less, it is less by an even number.

The number of negative roots of an equation f(x)=0 with real coefficients does not exceed the number of variations of signs in the sequence of the coefficients of f(-x) and if less, it is less by an even number.

Note: sign of zero in the sequence of coefficients will be taken as a continuation of the sign of preceeding nonvanishing coefficients.

**Example 2.15** If f(x)=2x3+7x2-2x-3 , express f(x-1) as a polynomial in x. Apply Descartes’ rule of signs to both the equations f(x)=0 and f(-x)=0 to determine the exact number of positive and negative roots of f(x)=0.

**»** By using method of synthetic division, f(x)=2(x+1)3+(x+1)2-10(x+1)+4. Let g(x)=f(x-1)=2x3+x2-10x+4. By Descartes’ Rule, g(x)=0 has exactly one negative root, say, c. Thus g(c)=f(c-1)=0; hence c-1(<0) is a negative root of f(x)=0. Since there are 2 variations of signs in the sequence of coefficients of f(-x)and since c-1 is a negative root of f(x)=0, f(x)=0 has two negative roots. Also, f(x)=0 has exactly one positive root ,by Descartes’ rule.

**Relations between roots and coefficients**

Let c1,…,cn be the roots of the equation a0xn+a1xn-1+…+an-1x+an=0. By factor theorem,

a0xn+a1xn-1+…+an-1x+an=a0(x-c1)(x-c2)…(x-cn).

Equating coefficients of like powers of x,a1=a0, a2=a0,….,an=a0 (-1)nc1c2…cn. Hence

=-,=,…, c1c2…cn=(-1)n.

**Example 2.16**Solve the equation 2x3-x2-18x+9=0 if two of the roots are equal in magnitude but opposite in signs.

**»** Let the roots be -a, a, b .Using relations between roots and coefficients, b=(-a)+a+b= and –a2b=-. Hence a2=9, that is, a=3. Hence the roots are 3,-3,.

**Example 2.17** Solve x3+6x2+11x+6=0 given that the roots are in A.P.

Symmetric functions of roots

A function f of two or more variables is symmetric if f remains unaltered by an interchange of any two of the variables of which f is a function. A symmetric function of the roots of a polynomial equation which is sum of a number of terms of the same type is represented by any one of its terms with a sigma notation before it: for example, if a,b,c be the roots of a cubic polynomial, then will stand for a2+b2+c2.

**Example 2.18** If a,b,c be the roots of x3+px2+qx+r=0, find the value of (1), (2), (3) ,(4),(5),(6),(7).

**»** (1)==p2-2q, (2) = -3abc=-pq+3r, (3)=-, (4) =-2abc, (5) =, (6) =, (7)=.

Transformations of equations

When a polynomial equation is given, it may be possible , without knowing the individual roots, to obtain a new equation whose roots are connected with those of the given equation by some assigned relation. The method of finding the new equation is said to be a transformation. Study of the transformed equation may throw some light on the nature of roots of the original equation.

1. Let c1,…,cn be the roots of a0xn+a1xn-1+…+an-1x+an=0; to obtain the equation whose roots are mc1,mc2,…,mcn. (m=-1 is an interesting case)

**»** Let d1=mc1. Since c1 is a root of a0xn+a1xn-1+…+an-1x+an=0, we have a0c1n+a1c1n-1+…+an-1c1+an=0. Replacing c1 by d1/m, we get a0d1n+ma1d1n-1+m2a2d1n-2+…+mn-1an-1d1+mnan=0. Thus the required equation is a0xn+ma1xn-1+…+mn-1an-1x+mnan=0.

1. Let c1,…,cn be the roots of a0xn+a1xn-1+…+an-1x+an=0 and let c1c2…cn0; to obtain the equation whose roots are .

**»** Let d1=. So c1=. Substituting in a0c1n+a1c1n-1+…+an-1c1+an=0, we get a0+a1d1+a2d12+…+an-1d1n-1+an d1n=0. Thus are the roots of anxn+an-1xn-1+…+a1x+a0=0.

1. Find the equation whose roots are the roots of f(x)=x4-8x2+8x+6=0, each diminished by 2.

**»** f(x)=(x-2)4+8(x-2)3+16(x-2)2+8(x-2)+6=0(by method of synthetic division). Undertaking the transformation y=x-2, the required equation is y4+8y3+16y2+8y+6=0

**Example 2.19** If a,b,c be the roots of the equation x3+qx+r=0, find the equation whose roots are (1) a(b+c),b(c+a),c(a+b), (2) a2+b2,b2+c2,c2+a2, (3) b+c-2a,c+a-2b,a+b-2c.

1. **»**a(b+c)==q-=q+. Thus the transformation is y=q+. Sustituting x= in x3+qx+r=0 and simplifying, we obtain the required equation.
2. **»**a2+b2=-c2=-2-c2=-2q-c2; hence the transformation is y=-2q-x2, or, x2=-(y+2q). The given equation can be written as x2(x2+q)2=r2; thus the transformed equation is (y+2q)(y+q)2=- r2.
3. **»**b+c-2a=-3a=-3a; the transformation is y=-3x.

**Example 2.20** Obtain the equation whose roots exceed the roots of x4+3x2+8x+3=0 by 1. Use Descartes’ Rule of signs to both the equations to find the exact number of real and complex roots of the given equation.

**»** Let f(x)= x4+3x2+8x+3=(x+1)4-4(x+1)3+9(x+1)2-2(x+1)-1 (by method of synthetic division). By Descartes’ rule, f(-x) has two variations of signs in its coefficients and hence f(x)=0 has either two negative roots or no negative roots; also f(x)=0 has no positive root(since there is no variation of signs in the sequence of coefficients of f). Undertaking the transformation y=x+1, f(x)=0 transforms to g(y)=y4-4y3+9y2-2y-1; considering g(-y), by Descrtes’ Rule, g(y)=0 has a negative root, say, a. Then f(x)=0 has a-1 as a negative root; the conclusion is f(x)=0 has two negative root, no positive root, does not have 0 as one of its roots and consequently exactly two complex conjugate roots (since coefficients of f are all real).

**Example 2.21** Find the equation whose roots are squares of the roots of the equation x4-x3+2x2-x+1=0 and use Descartes’rule of signs to the resulting equation to deduce that the given equation has no real root.

**»** The given equation is (x4+2x2+1)2=x2(x2+1)2; undertaking the transformation y=x2, the required equation is (y2+2y+1)2=y(y+1)2. Applying Descartes’ Rule to the transformed equation, it can be verified that the transformed equation has no nonnegative root and hence the original equation has no real root.

Cardan’s Method of solving a cubic equation

**Example 2.22** Solve the equation:x3-15x2-33x+847=0.

Step 1 To transform the equation into one which lacks the second degree term.

Let x=y+h. The transformed equation is y3+(3h-15)y2+(3h2-30h-33)y+(h3-15h2-33h+847)=0. Equating coefficient of y2 to zero, h=5. Thus the transformed equation is y3-108y+432=0 (\*)

Step 2 Cardan’s Method

Let a=u+iv be a solution of (\*). Then a3-108a+432=0 . also a3=u3+v3+3uv(u+v)= u3+v3+3uva; so a3-3uva-(u3+v3)=0. Comparing with the given equation, uv=36 and u3+v3=-432. Hence u3 and v3 are the roots of t2+432t+363=0. Hence u3=-216=v3. The three values of u are -6,-6w and -6w2, where w is an imaginary cube root of unity. Since uv=36, the corresponding values of v are -6,-6w2,-6w. Thus the roots of (\*) are -12,6,6 and thus the roots of the given equation are (using x=y+5) -7,11,11.

**Example 2.23** Solve the equation:x3+3x+1=0.

