**SEMESTER III MATHEMATICS HONOURS**

**ANALYSIS**

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**CHAPTER I**

**THE REAL NUMBER SYSTEM: AXIOMATIC DEFINITION**

THE SET N OF NATURAL NUMBERS

PEANO’S AXIOMS

Most familiar properties of N can be proved based on the following set of five axioms:

N1 1 belongs to N.

N2 If n belongs to N, then its *successor* n+1 belongs to N.

N3 1 is not the successor of any element of N

N4 If n and m in N have the same successor, then n=m.

N5 A subset of N which contains 1, and which contains n+1 whenever it contains n, is equal to N

Axiom N5 is the basis of mathematical induction. Let P1,P2,… be a list of statements that may or may not be true. The Principle of Mathematical Induction asserts that all the statements P1,P2,… are true provided (1)P1 is true, and (2) Pn+1is true whenever Pn is true.

**Example1.1** Show that , for all real numbers x.

Statement holds for n=1. Let hold for some natural number n. Then

Hence the result holds for all natural n.

The Principle of Mathematical Induction is equivalent to the following property of N:

**Well-Ordering Property of N**

Every nonempty subset of Natural Numbers has a least element.

THE SET Q OF RATIONAL NUMBERS

The space Q of rational numbers is a highly satisfactory algebraic system in which the binary operations addition, multiplication, subtraction and division (other than division by zero) can be fully studied. No system is perfect however and Q is inadequate in some ways.

The set Q of rational numbers is a very nice algebraic system until one tries to solve equations like x2=2. It turns out that no rational number satisfies this equation and there are good reasons to believe that some kind of number satisfies the equation. Consider a square with sides having length 1: if d represents the length of the diagonal, then d2=12+12=2. Apparently there is a positive length whose square is 2, which we write as . But is not a rational number, as we shall see. It is evident that there are lots of rationals and yet there are gaps in Q.

**DEFINITION 1.1** A number is called an **algebraic number** if it satisfies a polynomial equation anxn+an-1xn-1+…+a1x+a0 = 0 where the coefficients a0,a1,…,an are integers, an0 and n1.

Rational numbers are algebraic numbers since a rational number , m,n are integers and n0, satisfies the equation nx-m=0.Real numbers which are not algebraic are called **Transcendental Numbers.**

**Theorem 1.1** Suppose that a0,a1,…,an are integers and r is a rational number satisfying the polynomial equation anxn+an-1xn-1+…+a1x+a0=0, where n1, an and a00. Write r= , where p, q are integers having no common factors and q0. Then q divides an and p divides a0.

**Example1.2**  cannot represent a rational number.

**Answer** By theorem above, the only rational numbers that could possibly be solutions of x2-2=0 are 1 and . But none of the four numbers 1 and are solutions of the equation. Since represents a solution of x2=2, cannot represent a rational number.

We assume a familiarity with and understanding of Q as an algebraic system. However, in order to clarify exactly what we need to know about Q, we set down its basic axioms and some of its properties.

**Algebraic properties**

On Q are defined two binary operations addition and multiplication which satisfy following axioms:

For all a, b, c in Q, (1) a+(b+c)=(a+b)+c, a.(b.c)=(a.b).c (Associativity), (2) a+b=b+a, a.b=b.a (Commutativity), (3) 0,1, **01**, such that a+0=a,a.1=a(existence of identity for addition and multiplication), (4) aQ,(-a)Q such that a+(-a)=0; aQ,a-1Q such that a.a-1=1(existence of inverse under addition and multiplication), (5)a.(b+c)=a.b+a.c (disributivity of . over +)

A system that has more than one element and satisfies these five properties is called a **field**. The basic algebraic properties of Q can be proved solely on the basis of these field properties.

**Order properties**

On Q, there is defined a binary operation ‘’ (called order relation ) satisfying following properties:

1. For a,bQ, either ab or b, (2) if ab and ba, then a=b, (3) if ab and b, then ac, (4)if ab, then a+cb+c, (5) if a and 0c, then acbc.

A field satisfying above five properties is called an **ordered field**. Most of the algebraic and order properties of Q can be established for any ordered field.

**Theorem 1.2** In any field (F,+,.), following properties hold: For a, b, cF,

1. a+c=b+c ⇒ a=b, (2) a.0=0, (3) (-a).b=-(a.b), (4) ac = bc and c a=b, (5) ab=0a=0 or b=0,

**Theorem 1.3** In any ordered field (F,+,.,), following properties hold: For a, b, cF,

1. ab-b-a, (2)ab, c0bcac, (3) 0a and 0b 0ab, (4) 0a2, (5) 0<1

**Note** a<b iff ab and ab

**Proof**  (1)aba+[(-a)+(-b)]b+[(-a)+(-b)]-b-a(using associativity and commutativity)

1. ab, c0 ab,0-c (by (1)) a(-c)b(-c)

-acbcac, by (1).

1. Follows from order axiom (5).
2. By order axiom (1), a0 or 0a. If 0a, then 0a2, by (3). If a0, then 0-a, by (1).Thus 0(-a)2=a2.
3. 01 and 012=1.

**Examples of ordered field**

1. Let Q(t) ={p(t)/q(t): p, q are polynomials with coefficients in Q and q is not equal to identically zero}. Define addition and multiplication in Q(t) in the usual sense and S be the subset of elements of Q(t) such that the leading coefficients of the polynomials p and q have the same sign. Then Q(t) is an ordered field with S as the set of positive elements.
2. Let Q+ iQ={a+ ib: a, bQ}and define addition and multiplication in Q+iQ in the usual way. Then Q+iQ , endowed with these operations, is a field. But it is not possible to define an order in Q+iQ such that it turns out to be an ordered field: if it were an ordered field, nonzero square i2 must be positive ,-1 is negative and i2=-1 is a contradiction.

THE SET R OF REAL NUMBERS

The mathematical system on which we will do our analysis of properties of sets and functions will be the set R of all real numbers. The set R will include all rational numbers, all algebraic numbers,e and more. It will be a set having bijective correspondence with the set of all points on the number line: every real number will correspond to a point on the number line and every point on the number line correspond to a real number. In particular R will not have any ‘gaps’ unlike Q.

AXIOMATIC DEFINITION OF REAL NUMBER SYSTEM

Emphasizing similarity between Q and R, we assume:

R is an ordered field.

Below we shall see how to distinguish between Q and R: we shall give a ‘gap filling’ axiom to distinguish Q and R.

THE COMPLETENESS AXIOM

The proofs of many of the basic theorems of calculus—existence of maxima, minima, the intermediate value theorem, Rolle’s Theorem, the Mean Value Theorem etc. depend heavily on the completeness property of real numbers. The Completeness Axiom ensures that R has no ‘gaps’. R is defined as an ordered field which satisfies the completeness axiom whereas Q does not satisfy the axiom. It has far reaching consequence and every significant result of Calculus relies on it. **There are many equivalent ways to formulate the completeness property for R: Least Upper Bound property, Nested Interval Property, Convergence of Cauchy sequences property etc.** We start with LUB Property.

**DEFINITION 1.2** Let SR. If a real number M satisfies sM for all sS, then M is called an upper bound of S and S is called bounded above. If a real number m satisfies ms for all sS, then m is called a lower bound of S and S is called bounded below. S is bounded iff S is both bounded above and bounded below. Thus S is bounded iff there exist real numbers m and M such that S[m,M].

**Example1.3** The maximum of a set, if it exists, is an upper bound of the set. Let a, bR, a<b. b is an upper bound of each of the sets [a,b], [a,b),(a,b] and (a,b). Note that every number larger than b is also an upper bound for each of these sets. Any non-positive real number is a lower bound for {rQ: 0.

**DEFINITION 1.3** Let SR. If a real number M satisfies (1) M is an upper bound of S and (2)no real number less than M is an upper bound of S, then M is the least upper bound of S or supremum of S, written as l.u.b. S or sup S. Dually, a real number m satisfying (1) m is a lower bound of S and (2)no real number greater than m is a lower bound of S, is the greatest lower bound of S or infimum of S, written as g.l.b.S or inf S.

The maximum (minimum) of a nonempty subset S of real numbers, if it exist, is the (prove uniqueness of sup, if it exists!) supremum (infimum respectively) of S. But a nonempty subset of real numbers (e.g. (0,1)) may not possess maximum but may possess lub which does not belong to the set. If the supremum of a bounded above subset M of real numbers belong to the set M, then M must be the maximum element of the set.

**Completeness Axiom for R**

Every nonempty subset S of R that has an upper bound in R has a least upper bound in R.

**Corrolary**  Every nonempty subset S of R that is bounded below has a greatest lower bound in R, denoted by glb S or inf S .

**Example1.4** Inf {1/n: n natural}=0. Let A= {1/n: n natural}. A is bounded below by zero, so A has the greatest lower bound a in R. Clearly 0a, for all a in A. On the other hand, a1/(2n) for all positive integer n(since a is a lower bound for A), so 2a is a lower bound of A; thus 2aa; therefore a0 and hence a=0.

**Note The ordered field Q of rational numbers is not complete.** Let A={rQ: r>0and r2<2} and B={rQ: r>0 and r2>2}. 1A, 2B ; hence A,B. If rQ and r2, then r24>2, so rA; thus r<2 for all rA; thus A is bounded above. We next verify that A has no largest element: given any element r of A, we try to find a positive integer n such that r+A, that is, (r+1/n)2<2 which is equivalent to the condition n(2-r2)>2r+1/n. Since 2-r2>0, the Archimedean property yields a positive integer n such that n(2-r2)>2r+12r+1/n. Thus A has no largest element.

Next we show B has no smallest element. Let rB; it suffices to find a positive integer n such that r-1/n>0 and (r-1/n)2>2, or equivalently, nr>1 and n(r2-2)>2r-1/n. Since r>0 and r2-2>0, the Archimedean property yields, possibly on taking maximum, a positive integer n such that both nr>1 and n(r2-2)>2r>2r-1/n; thus r-1/n is an element of B smaller than the element r of B.

Finally, we assert that A has no least upper bound in Q. Assume, to the contrary, that lub A=tQ. t since ∉Q. t>0 since 1A. If t2<2, then tA but then t would be the largest element of A, contradiction. If t2>2, then tB; since B has no smallest element, there exists sB with s<t=lub A; thus s is not an upper bound of A; thus there exists rA such that s<r, but then s2<r2<2, contradiction.

It can be proved that **(1) there exists a complete ordered field** and **(2) any two complete ordered fields are isomorphic** [**that is, if F and G are complete ordered fields, then there exists bijective function :FG such that (1) (a+b)=(a)+(b), (2)(ab)=(a)(b) and (3) ab in F implies (a) (b) for all a,b in F**].

In this sense, we talk about THE Complete Ordered Field of Real Numbers.

In an arbitrary ordered field, there may exist elements a>0 and b>0 such that a<1/n and n<b for all positive integer n; but such strange elements cannot exist in R because of Completeness Axiom .

**DEFINITION 1.4** An ordered field F is Archimedean if for x, yF, x>0, there exists positive integer n such that nx>y holds.

**Archimedean Property of R**

**Theorem 1.4**  The ordered field R of real numbers is Archimedean.

**Proof** Let x, y be real numbers with x>0. If y0, then 1x>y. Let y>0. Since inf {1/n: nN}=0 and x/y>0, there exists positive integer n such that 1/n<x/y, hence n x>y.

The Archimedean property tells us that given enough time, one can empty a large bathtub with a small spoon.

**Corrolary** Taking b=1, we see that for any a>0, there exist positive integer n such that 1/n<a; taking a=1, we see that for every real number b, there exist positive integer n such that n>b (that is , the set of all positive integer is unbounded above).

**Example1.5** Q is Archimedean.

**Proof** If x>0,y=0, then 1.x>0=y. If x>0, y<0, then 1.x=x>0>y. Hence the essential case is x, yQ with x>0 , y>0. Let x=a/b, y=c/d with a, b, c, d positive integers. Let n=1+bcN. Then nad=(1+bc)ad>bc, that is, dividing by abcd>0, n x>y.

The ordered field Q(t) considered above does **not** satisfy the Archimedean property: for the elements 1(>0) and t in Q(t), there does not exist any positive integer n such that n.1>t, for all rational t.

**Example1.6** N is unbounded above.

**Proof** If possible, let N be a bounded above subset of R. Then there exists aR such that na, for all natural n. Obviously, a>0. By Archimedean property, corresponding to x=1>0,y=a, there exists natural m such that m.1>a, contradiction.

Following is another result that holds for R but may not hold for an arbitrary ordered field:

**Denseness of Q in R**

**Theorem 1.5** If a, b are real numbers and a<b, then there exist a rational number r such that a<r<b.

**Proof Case 1 0<a<b**

By Archimedean property, there exists natural k such that k(b-a)>1. Thus <b-a. Let A={nN: n.>a}. Applying Archimedean property to the pair , we see A. By well ordering principle of N, A has a least element n0. Thus n0A and n0-1∉A. Thus and (n0-1)a. Hence a+<a+(b-a)=b. Thus a<<b and is rational.

**Case2** a.

By Archimedean property, there exists natural k such that<b[ considering the pair (b,1)]. Thus a<<b, rational.

**Case 3** a<b<0

-a>-b. By discussion above, there exists rational r such that –b<r<-a. Thus a<-r<b, -r rational.

**Denseness of Set of Irrationals in R**

**Theorem 1.6** If a,b are real numbers and a<b, then there exist an irrational number i such that a<i<b.

**Proof** By previous result, there exists nonzero rational number r satisfying . Thus r is an irrational between a and b.

**Example1.7** For each of the following sets, verify whether the set is bounded above and/or below and in case they are, find the sup and/or inf of the sets:

1. , (2) , (3) , (4)

**Example1.8** Let S be a nonempty subset of R that is bounded above. Prove that if sup S belongs to S, then sup S= max S.

**Example1.9** Let S. Prove that inf Sy about S if inf S=sup S?

**Example1.10** Let S be a nonempty subset of R which is bounded below. Prove that inf S=-sup{-s/sS}.

**Example1.11** Let SR and suppose that supS=sS and u does not belong to S. Prove that sup(S= sup{s,u}.

**Example1.12** Let S and T be nonempty bounded subsets of R. Prove that if S, then inf T inf Ssup S. Prove also that sup(ST)= max{sup S, sup T}.

**Example1.13** Prove that if a>0, then there exists n1/n<a<n.

**Example1.14** Show that sup{r: r<a}=a for each a

**Finite and infinite set**

Finite and infinite sets are defined below without reference to the set of all positive integers; of course, eventually the set of positive integers must be brought in. The main objective is to rehearse some of the most frequently used arguments in discussions involving finiteness and infiniteness.

**DEFINITION 1.5** A set E is infinite if there exists an injection EE that is not surjection (that is, if E is in bijective correspondence with a proper subset of itself). A set that is not infinite is said to be finite.

**Example1.14(1)** The set N of all positive integers is infinite: nn+1 is the required injection: N is in bijective correspondence with N-{1}.

(2) The null set is finite ‘by default’ (null set is not the domain of a function); for a non-null set E, E is finite iff every injection EE is a surjection.

(3) Singleton {c} is finite since f(c)=c is the only available function.

**Note** Every superset of an infinite set is infinite; every subset of a finite set is finite. Let E be an infinite set, EF, EF: by assumption , there exists an injection f:EE that is not surjective. The function g: F defined by g(x)=f(x), if xE and =x, if xF-E is injective but not surjective.Thus F is infinite.

**Theorem 1.7** Let f: EF. The following result holds:

1. If f is bijective and E is infinite, then F is infinite
2. If f is bijective and E is finite, then F is finite
3. If f is injective and E is infinite, then F is infinite
4. If f is surjective and E is finite, then F is finite

**Proof** (1) Let E be infinite and f be bijective. Let g: EE be an injection that is not a surjection.The mapping f0g0f-1: FF is an injection (composition of injections) but not surjective: if it were, then f-10(f0g0f-1)0f=g would be surjective.

(2) If f: EF is bijective, then so is f-1:FE; if F were infinite, then by (1), E would be infinite, contradiction.

(3) If E is infinite and f: EF is an injection, then f defines a bijection Ef(E), therefore f(E) is infinite, by (1). Hence, by preceeding note, F is infinite.

(4) Let f: EF be surjective. Using axiom of choice, there exists an injection g: FE; if F were infinite, then by (3), E would be infinite, contradiction.

**Theorem 1.8** A setE is infinite iff there exists an injection NE.

**Proof** If there exists an injection NE, since N is infinite(example 1.1), E is infinite.

Conversely, assuming E is infinite , let f:EE be an injection that is not a surjecion. Choose zE-f(E). Define g: NE by g(n)=fn(z), where f1=f and fn+1= fn0f for natural n. We assert that g is injective. If possible, let m, nN , m<n, g(m)=g(n). Let p=n-m; then g(n)=fm+p(z). Then fm(z)=g(m)=g(n)=fm+p(z) and since f is injective, z=fp(z)f(E), contradiction.

**Notation** for each positive integer n, Pn={1,…,n}. The next target: A nonempty set E is finite iff it is bijective with some Pn.

**Lemma 1.1** If S is a nonempty subset of N with no largest element, then S is infinite.

**Proof** For each aS, let S(a)={kS: k>a}. By assumption, S(a) . Define f: SS as follows: for each aS, let f(a) be the smallest element of S(a). In particular, f(a)>a for all aS. We show that S is infinite by showing that f is injective but not surjective. If a, bS, a<b, then bS(a), therefore, f(a)b<f(b); hence f is injecive. f is not surjective, for , if z is the smallest element of S(existence of z is guaranteed by Well-Ordering Principle of N), then za<f(a), for all aS, therefore z does not belong to f(S).

**Lemma 1.2** If A is a finite subset of N, then N-A has no largest element.

**Proof** Let B=N-A. If possible, let m be a largest element of B; then BPm={1,2,…,m}. Then N- PmN-B and N- Pm ={k:k>m} is infinite since kk+1 is an injection under which m+1 has no preimage; hence N-B=A is infinite, contradiction.

**Corrolary** If A is finite subset of N, then N-A is infinite.

**Lemma 1.3** N is not the union of two finite sets.

**Proof** Let N=AB with A finite ; then N-AB and N-A is infinite; hence B is infinite.

**Lemma 1.4** If f: EF and A is a finite subset of E, then f(A) is finite.

**Proof** The restriction of f to A defines a surjection Af(A). Result follows from Theorem 1.1,(4).

**Lemma 1.5** If A and B are finite sets, then AB is finite.

**Proof** let E=AB and assume to the contrary that E is infinite. By Theorem 1.2, there exists an injection f:NE and hence there exists a surjection g:EN. Then N=g(E)=g(AB)=g(A)g(B) is union of two finite sets, contradiction, by Lemma 1.3.

**Lemma 1.6** For every positive integer n, Pn is finite.

**Proof** P1={1} is finite (Example 1.3). Let Pn be finite. Then Pn+1=Pn{n+1} is finite , by Lemma 1.5. Thus result holds by induction.

**Lemma 1.7** Let m, nN. If there exists a bijection f: PnPm, then m=n.

**Proof**  we can suppose nm;if mn, we may consider the bijective function f-1. Then PmPn. Let i:PmPn be the iclusion mapping. The composite function i0f:PnPn is injective and , since Pn is finite(Lemma 1.6), is surjective and hence bijective. Hence (i0f)0f-1=i is also bijective ; so Pm=Pn and hence m=n(note: k is the largest element of Pk).

**Theorem 1.9** A nonempty set is finite iff it is bijective with some Pn.

**Proof** If f:EPn is bijective, then since Pn is finite(Lemma 1.6), E is finite (Theorem 1.1,(2)).

Arguing contrapositively, if E is a nonempty set that is not bijective with any Pn, we must show E is infinite. By Theorem 1.2, it suffices to find an injection NE, in other words, a sequence (xn) in E such that nxn is injective (that is, the xn are pairwise distinct). An informal ‘recursive’ argument for producing such a sequence is as follows: choose x1E and let E1={x1}. Let n1 and assume distinct points x1,…,xn of E have already been chosen. Let En={x1,…,xn}. Then ixi is a bijection PnEn, so by hypothesis EnE; choose xn+1E-En.

**DEFINITION 1.6** If a set E is bijective with some Pn, then n is unique(Lemma 1.7); n is called cardinality of E: we write n=card E. In particular, card Pn=n for all nN.

Countable and Uncountable Sets

**DEFINITION 1.7** A set E is at most countable if either E= or there exists a surjection NE. If E is not at most countable, then E is uncountable.

**Remarks (1)** If E is at most countable (uncountable) and if f: EF is bijective, then F is at most countable (uncountable resp.). **(2)** Every subset of an at most countable set is at most countable(hence every superset of an uncountable set is uncountable) **(3)**Every finite set is at most countable.

**Proof** (2) Suppose f:NE is surjective and A be a nonempty subset of E: we have to find a surjection NA. Choose aA and g:NA be the function such that g(n)=f(n), if f(n)A, and g(n)=a, if f(n)∉A. For every xA, there exists an n with f(n)=x, therefore g(n)=x; thus g is surjective.

(3) A nonempty finite set is bijective with some Pn, hence is at most countable.

**Lemma 1.8** Every infinite subset of N is bijective with N.

**Proof** Let A be an infinite subset of N; we must construct a bijection f:NA. Define f(1) to be the smallest element of A and, recursively, f(n+1) to be the smallest element of A-{f(1),…,f(n)}. f is strictly increasing [ f(2)f(1), since f(1) is the smallest element of A,f(2)A, f(2)f(1); in general , there are no elements of A between f(n) and f(n+1) for any natural n, whence f(n+1)>f(n)]. Also an easy induction shows that f(n)>n for every n. Thus f is injective. Next we show f is surjective. Assume, to the contrary,that A-f(N) contains some element k. In particular, kA-{f(1),…,f(k)}, so kf(k+1) by the minimality of f(k+1), whence the absurdity k+1f(k+1)k.

**Theorem 1.10** A set E is at most countable iff either (1) E is bijective with N or (2) E is bijective with Pn={1,…,n} for some positive integer n or (3) E=.

**Proof** That a set satisfying (1) and (3) is at most countable is obvious from definition and a set satisfying (2) is at most countable by remark (1) and (2) following definition 1.2.

‘only If’ Suppose E and f:NE be surjective; we have to show that E satisfies (1) or (2). If E is finite, we are done by Theorem 1.3. Suppose E is infinite. For every xE, let Ax=f-1{x} (Ax, since f is surjective),let g(x) be the smallest element of Ax; this defines a function g:EN , injective since AxAy= when xy. S; since E is infinite, so is g(E); thus g(E) is bijective with N by Lemma 1.8, thus so is E.

**DEFINITION 1.8** A set is countable iff it is bijective with N.

The following result is a ready source of uncountabsle sets:

**Theorem 1.11** If E is a set and P(E) is its power set, then there does not exist a surjective mapping E P(E).

**Proof** Assume , to the contrary, that there exists a set E that admits a surjective mapping f:E P(E) . Clearly, E(there cannot be a surjection {}).Let A={xE:x∉f(x)}. Since f is surjective, A=f(a) for some aE. Either (1) aA or (2) a∉A. If aA=f(a), then (by definition of A), a∉f(a). If a∉f(a)=A, then af(a)=A. Hence the result.

**Corrolary**  P(N)is uncountable, since there is no surjection N P(N).

**Theorem 1.12** NXN is bijective with N.

**Proof** f: NXNN defined by f(m,n)=2m3n is injective, so NXN is bijective with its range A=f(NXN); but A is infinite (the mapping m2m is an injection NA), therefore A is bijective with N(Lemma 1.8).

**Corrolary**  If E and F are countable, then so is E X F.

**Proof** Let g:NE and h:NF be the corresponding surjections; the mapping NXNEXF defined by (m,n)(g(m),h(n))is surjective, and by Theorem 1.6, there exists a surjection NNXN, so composition yields a surjection NEXF.

**Corrolary** The field Q of rational numbers is countable.

**Proof** If S={m/n: m, nN} (the set of positive rational numbers), then NS and the mapping (m,n)m/n is a surjection NXNS. Since NXN is bijective with N (Theorem 1.6), S is at most countable. By Theorem 1.4, since S is neither finite nor , S is countable ; let f:NS be a bijection. Let rn=f(n), n natural. Then the sequence 0, r1,-r1,r2,-r2,… exhausts Q and defines a bijection NQ.

**Corrolary** The field R of real numbers is uncountable.

**Proof** By remarks following definition 1.2,it suffices to show that [0,1] is uncountable. For this, we need to show that that every mapping f: N[0,1] fails to be surjective. For any closed interval [a,b], let c and d be its points of trisection(a< c<d<b) and call [a,c],[c,d], [d,b] the closed thirds of [a,b]. Define recursively a decreasing sequence I1⫆I2⫆… of non-degenerate closed sub-intervls of [0,1] as follows: I1 is a closed third of [0,1] such that f(1)I1(to be definite, we may choose I1 to be the left most third with this property) and, recursively, In+1 is a closed third of In such that f(n+1)In+1.By Nested Interval Property, contains a point x. For every n, we have xIn, therefore, f(n)x , thus x fails to belong to the range of f.

**Alternative proof of uncountability of R**

It is sufficient to prove that [0,1) is uncountable. If possible, let [0,1) = {a1,a2,…}. Let the decimal representation of elements an be as follows(unless we allow recurring use of 9, the representation is unique: compare 0.999…=1.000…):

a1=0.a11 a12….

a2=0.a21 a22…

……

Where aij=0,1,…,9 (recurring use of 9 excluded)

Then the decimal number a=0.x1 x2… where xn=1, if ann1 and xn=2, if ann=1, belongs to [0,1) but a{ a1,a2,…}. Hence the proof.

NOTE Q is countable but not complete; R is complete but not countable.

**CHAPTER II**

**REVISION OF METRIC SPACES**

**Example 2.1**On Rn, define d1()=, d2(=max{}, d3(= ,d4()= (1p). d1,d2,d3 , d4 are metrics on Rn.

**Example 2.2**Let be the set of all bounded sequences in R or C and define }. Then () is a metric space.

**Example 2.3**Let B[a, b] be the set of all real valued functions defined and bounded on [a, b]. Define (x,y)= sup{. (B[a,b],) is a metric space.

**Example 2.4**Let C[a,b] be the set of all real valued functions defined and bounded on [a,b]. Define (x,y)= max{. (C[a,b],) is a metric space.

**Example 2.5**For any nonempty set X, define d(x, y)=1, if xy and d(x,y)=0,if x=y. (X,d) is a MS, called a discrete MS.

**Open Spheres, Closed Spheres**

Let (X, d) be a MS, xX and r>0. The open ball with centre x and radius r , denoted by B(x, r), is the set {yX: d(x,y)<r}. The closed ball with centre x and radius r , denoted by B[x, r], is the set {yX: d(x,y)r}. In the MS Ru of real numbers with usual metric d(x,y)=, B(x,r)=(x-r,x+r) and B[x,r]=[x-r,x+r]; conversely, every open interval (a, b)=B( is an open ball. In R with discrete metric, {a}=B(a,1/2) is an open ball, for every real a.

**Open sets, Closed sets**

**Definition 2.1** Let (X,d) be a MS and AX. A is open in X iff either (1) A= or (2) A and for each aA, there exists ra>0 such that B(a, ra)A. A subset C of X is closed in X iff X-C is open in X.

**Example 2.1** (0,1) is open in Ru but [0,1) is not. N, Z, Q are not open in Ru. Set of rationals and set of irrationals are not open in R. In the MS of real numbers with discrete metric, every subset is open. R is closed in R since R-R= is open. In a discrete metric space, every subset is clopen ( closed and open).

**Note** Every open ball in a MS (X,d) is open in (X,d), converse need not hold. A subset A of X is open iff A is union of open balls. Arbitrary union and finite intersection of open sets in a MS is open; though arbitrary intersection of open sets need not be open: consider ={0} is not open in Ru.

Every closed ball is a closed set. Arbitrary intersection and finite union of closed sets is closed; arbitrary union of closed sets need not be closed [ consider in Ru]. Any finite subset in a MS is closed in the MS.

**Interior Point, Limit Point, Interior and Closure**

**Definition 2.2** Let (X,d) be a MS ,AX, xA. x is an interior point of A iff there exists r>0 such that B(a, r)A. Interior of A, denoted by A0, is the set of all interior points of A. A is open in (X,d) iff A=A0. In general, A0 is the largest open set of X contained in A. For two subsets A,B of X, (1) AB implies A0B0, (2)(AB)0=A0B0, (3) A0 B0 (AB)0 [counterexample: In Ru, take A=[0,1], B=[1,2] or, A=Q,B=R-Q].

**Definition2.3** Let (X,d) be a MS ,AX. x(X) is a limit point of A if each open ball centred at x contains at least one point of A other than x, that is, [B(x,r)-{x}]. The set of all limit points of A, denoted by A/, is called the derived set of A.

**Theorem 2.1** Let (X, d) be a MS and AX. A is closed in X iff A/A.

**Proof** Let A be closed. Let xA/-A. Thus xX-A, an open set in X. Thus there exists r>0 such that B(x,r)X-A. Hence [B(x,r)-{x}], contradicting xA/. Hence A/A.

Conversely, let A/A. Let xX-A. Then xA and xA/. Thus there exists r>0 such that B(x,r)X-A. Hence X-A is open and A is closed.

**Note** If x is a limit point of AX, then any open ball containing x contains infinitely many distinct points of A. If not, let x1,…,xn be the only points(besides possibly x) of A that are contained in some ball B(y,r)containing x. Let s=min {d(x,x1),…,d(x,x­n),d(x,y)}. Then [B(x,s)-{x}]A=, contradiction.

On the other hand, if xA-A/, then there exists r>0 such that B(x,r)A={x}; such a point x is an isolated point of A in X. Thus if X be a MS, xAX, then x is either a limit point of A or an isolated point of A. In Ru, 0 is a limit point of A={0}{1/n:nN} but all other points of A are isolated points of A.

**Definition 2.4** Let (X,d) be a MS and AX. Closure of A, denoted by , is defined to be equal to A.

**Theorem 2.2** Let (X,d) be a MS and AX. Then (1) is a closed set, (2) A is closed iff A=, (3) is the smallest closed set containing A, (4) is the intersection of all closed subsets of X containing A, (4) AB implies , (5) , (6) .

**Proof**

1. Let x()/ and let r>0 be arbitrary. Since x is a limit point of , there exists y such that d(x,y)< r/2. Since y either belongs to A or is a limit point of A or both, there exists zA such that d(y,z)< r/2. Thus d(x,z)<r. If x=zA, proof is over. If xz, then by arbitrariness of r, xA/. Combining the two cases, ()/, that is , is closed.
2. A is closed iff A/ iff A=A A/=.
3. If B is any closed set containing A, then AB implies A/B/B/==B and hence =AA/B.Thus.
4. Follows from (1) and (3).
5. Since is a closed set containing AB, by(3), . Conversely, let x. If xAAB, nothing remains to prove. Let xA/-A. For every r>0, [B(x,r)-{x}]A and hence [B(x,r)-{x}]A; thus x(A)/. Similarly,. Thus .
6. Try yourself; Counterexample: take (0,1) and (1,2) in Ru.

**Theorem 2.3**  Let A be a nonempty set of real numbers which is bounded above and let y=sup A. Then y. If A is closed, then yA.

**Proof** If yA , then y. Assume y∉ A. By definition of supremum, for every r>0,there exists xA such that y-r<x<y<y+r; thus yA/. Hence y.

**Definition 2.5** Let (X,d)be a MS, xAX. x is a boundary point of A iff for every r>0, B(x,r)A and B(x,r)(X-A).

**DISTANCE BETWEEN SUBSETS; DIAMETER OF SUBSETS**

**Definition2.6** Let (X,d) be a MS, A,B be nonempty subsets of X. The distance of a point xX from A , denoted by d(x,A), is defined by d(x,A)=inf{d(x,y):yA}[ Note that the subset {d(x,y):yA} of real numbers is bounded below by zero, hence inf exists as real number]. Distance between the subsets A and B, denoted by d(A,B), is defined by d(A,B)=inf{d(a,b):aA, bB}. The diameter of A, denoted by d(A), is defined by d(A)=sup{d(x,y): x, yA}.

**Theorem 2.4**  Let (X,d) be a MS, A,B be nonempty subsets of X. Then: (1) x iff d({x},A)=0, (2) d(,)=d(A,B), (3) d()=d(A).

**Proof**

1. If xA, then 0d(x,x)=0, so that d({x},A)=0. Let xA/-A. Corresponding to 1/n>0, there exists xnA such that d(xn,x)<1/n. Thus 0d(x,xn)<1/n; by Squeeze Theorem on sequence, d({x},A)=0. Conversely, if d({x},A)= inf{d(x,y):yA}=0, then for every positive r, r is not the infimum of {d(x,y):yA}; hence, there exists yrA such that d(x, yr)<r . Thus [B(x, r)-{x}]ASo .
2. Try yourself.
3. Since A, d(A)d(). Conversely, let x1,y1. Let r>0 be given. Corresponding to chosen r>0, there exist x, y in A such that d(x, x1)<r/2, d(y,y1)<r/2. Then d(x1, y1)<r+ d(x, y)r+d(A). Hence d()r+d(A). By arbitrariness of r>0, d()d(A). Hence the result.

**Note** Let X= ,A={1,1/3,1/5,…} and B={1/2,1/4,…} . A and B are disjoint and closed subsets of X (considered as a subspace of Ru) though d(A,B)=0 [ 0d(=, for every natural n].

**Definition 2.7** Let (X,d) be a MS, AX. A is bounded iff d(A)<; otherwise, A is unbounded. In particular, X is bounded iff d(X)<.

**Note** Let (X, d) be a MS. Define d1,d2 on X X X by: d1(x,y)= and d2(x,y)= min{1,d(x,y)}. Then d1, d2 are equivalent metrics to d; (X, d) may or may not be bounded though (X,d1) and (X,d2) are bounded[Note: d1(X)<1 and d2(X)<1]. Note that every set in a discrete MS is bounded.

Subspace of a Metric Space

**Definition 2.8** Let (X, d) be a MS and YX. Define dY: Y X YR by dY(x, y)=d(x, y) for all x, yY. It can be verified that (Y, dY) is a MS and it is called a subspace of the MS (X,d).

Thus the set P[a, b] of all polynomials form a metric subspace of the MS (C[a, b], of all continuous functions on [a, b] under the metric (f,g)=sup}.

**Note** If Y is a subspace of a MS X, then a subset of Y which is open in Y need not necesarily be open in X: If X=Ru, Y=[0,1], then [0,1/2) is open in Y but not in X; if Y=(0,1), then the set (0,1/2] is closed in Y but not in Ru.

**Lemma 2.1** Let(Y, dY) be a subspace of a MS (X, d), let aY and r>0. Then BY(a, r)=YB(a,r) where BY(a, r)={yY: d(a,y)<r} and B(a,r)={xX:d(a,x)<r}.

**Theorem 2.5** Let(Y, dY) be a subspace of a MS (X, d) and AY. Then (1) A is open in Y iff there exists an open set O in X such that A=OY, (2) A is closed in Y iff there exists closed set C in X such that A=CY, (3) ClYA=ClX AY.

**Proof**

1. Let A=OY, where O is open in X. Let aA. Then aO,an open set in X. Thus there exists r>0 such that B(a,r)O. Now aB(a,r)Y=BY(a,r)Y. Thus an arbitrary point of A is an interior point of A in Y; hence A is open in Y.

Conversely, let A be open in Y: then A= where ra is a suitable positive real numbers that, in general, varies with a. Thus A== =Y = OY, where O= is open in X.

1. A is closed in Y iff Y-A is open in Y iff Y-A=OY, where O is open in X

iff A=Y(Y-O)=Y-YO=(XY)-(OY)=(X-O)Y=CY, where C is closed in X.

1. A ClXAY and ClXAY is closed in Y; hence ClYA ClXAY. Conversely, ClYA is closed in Y; hence by (2), ClYA=CY for some C closed in X and A ClYAC. Thus ClXA; hence ClXAYY= ClYA; hence the equality.

**Note** If A is open(closed) in Y and Y is open(closed) in X, then A is open(closed) in X.

**Dense Sets and Separable Spaces**

**Definition2.9** Let (X,d) be a MS and AX. A is dense (or everywhere dense) in X iff =X.

Example

1. In a discrete MS X, the only dense set is X itself since every singleton is open.
2. Q is dense in Ru.

**Theorem 2.6** Let (X,d) be a MS and AX. Then the following are equivalent: (1) A is dense in X, (2) The only open set disjoint from A is , (3) A intersects every open sphere.

**Proof**

(3)⇒(1) Let B(x,r)A for all xX and for r>0. Let a X. For every r>0, A; hence a. Thus =X.

(1) ⇒(3) Let =X and ,if possible, let B(a,r)A=, for some aA and some r>0. Then a does not belong to , contradicting =X.

(1) ⇒(2) Let A be dense in X and , if possible, let U ( be open in X such that UA=. Let aU. Then there exists B(a,r)U such that B(a,r)A=;hence a does not belong to , contradicting =X.

(2) ⇒(1): try yourself.

**Definition2.10**  A MS is separable iff it has a countable dense subset.

Example

(1)Ru is separable but R under discrete metric is not separable since the only dense set in R is R, which is uncountable.

(2) (C[a,b], , where (f,g)=sup{}, is separable. Let fC[a,b]. By Weierstrass Approximation Theorem, there exists a sequence (pn) of polynomials with real coefficients such that (f,pn)0 as n. But any polynomial with real coefficients can be uniformly approximated by a sequence of polynomials with rational coefficients since Q is dense in R.

[Let >0. Let f(x)=a0+a1x+…+anxn, ai real, i=1,…,n. By density of Q in R, there exists sequences {bij} of rational numbers converging to ai, i=1,…,n. Thus {bij}ai, i=1,…,n. Thus there exist natural m1,…,mn such that nmi implies <, for i=0,…,n. Let M=max{ m1,…,mn}. Then nM implies (f,b0j+b1jx+…+bnjxn) ]

Thus there exists a sequence (q­n) of polynomials with rational coefficients such that 1/n for all t in [a,b], that is ,(pn,qn)<1/n for all natural n. Hence (x,qn)(x,pn)+(pn,qn)0 as n. Thus the countable set of polynomials with rational coefficients is dense in C[a,b].

**Note** Concept of distance between two real numbers is essential for the discussion of limit, continuity of real valued function of real variable. In Sem I, we have discussed those concepts in

**Ru**, the MS of real numbers under the usual metric: d(x,y)=. These concepts can be discussed in the general setting of an arbitrary MS (X,d). **Results derived from the general setting of a MS can be applied, in particular , to Ru; but the converse may not hold**. **We must note that Ru is a much richer structure than an arbitrary MS; further the metric in Ru arise from the order structure of R. Thus results in Ru and (X,d) should be compared carefully. Note , in particular, that LUB, NIP and CSC are all equivalent form of completeness in Ru but in (X,d) CSC form of completeness prevails though LUB and NIP form of completeness cannot be framed in (X,d).**

**CHAPTER III**

**Convergence and Completeness in Metric Spaces**

**Definition 3.1** Let (X,d) be a MS and (xn) be a sequence of elements of X. (xn) converges to an element x of X in (X,d), written lim (xn)=x, iff for every >0, there exists positive integer m such that nm implies d(xn, x)<.

**Note** The property of convergence of a sequence in a MS(X,d) is not inherent in a sequence but depends on the set X and on the metric d. The sequence is convergent in [0,1] but not in (0,1)(both considered as subspaces of Ru). The sequence (xn) in C[0,1] defined by xn(t)=e-nt, 0, is convergent to x(t)=0, 0, in the metric d1[ d1(x,y)=] but not in the metric [].

**Theorem 3.1** Every convergent sequence is bounded.

**Proof** Let A={xn: n natural}. Let lim(xn)=x in (X,d). Corresponding to =1, there exists natural number m such that for nm, d(xn,x)<1 holds. Let M=max{d(x1,x),…,d(xm-1,x),1}. Then d(xn,x)M, for all natural number n. Thus d(xp,xq) d(xp,x)+d(x,xq)2M, for any natural p,q. Thus d(A)=sup d(xp,xq) 2M. Hence the result.

**Theorem 3.2** Every convergent sequence in a MS has unique limit.

**Proof** Let lim(xn)=L and lim(xn)=M. Corrseponding to >0 chosen, there exist natural number p,q such that np implies d(xn,L)< and nq implies d(xn,M)<. Let m = max{p,q}. Then nm ⇒0d(L,M)d(L,xm)+d(xm,M)<2. By arbitrariness of >0, d(L,M)=0 and hence L=M.

**Theorem 3.3** Let (X,d) be a MS and AX. Then x(X) is a limit point of A iff there exists a sequence (xn) in A such that xnx for all n and lim(xn)=x.

**Proof** Let x be a limit point of A. For every natural n, choose xn[B(x,)-{x}]; such xn exists corresponding to every natural n since x is a limit point of A. Since 0d(xn, x)<, lim(xn)=x, xnx for all n.

Conversely, let lim(xn)=x, xnx,xn A.for all n. For every >0, there exists natural m such that d(xn,x)< for every natural nm. Thus xm [B(x,)-{x}]A. Hence [B(x,)-{x}]A for every . Thus x is a limit point of A.

**Note**

1. A subset A in a MS (X,d) is closed iff every convergent sequence(in (X,d)) in A has its limit in A.
2. Let (xn) be a sequence in a MS (X,d) convergent to xX. Let A be the range set of the sequence (xn). If A is finite, then xn=x for infinitely many n. If A is infinite, then x is a limit point of A.

Example3.1A sequence in a discrete MS is convergent iff the sequence is eventually constant(that is, with possible exception of a finite number of terms, all terms are equal).

**Cauchy Sequences**

**Definition 3.2** A sequence (xn) in a MS is Cauchy iff for every >0, there exists positive integer m() such that for all positive integer p,qm, d(xp,xq)<. Equivalently, (xn) is Cauchy iff for every >0, there exists positive integer m() such that for all positive integer nm and for all positive integer p , d(xn+p,xn)< holds.

**Note** Let (X,d) be a MS, YX, (yn) be a sequence in Y. (yn) is CS(Cauchy Sequence) in X iff (yn) is Cauchy in Y(considered as a subspace of X)

**Theorem 3.4** Every Cauchy sequence is bounded.

**Proof** Let (xn) be a CS in a MS (X,d). Corresponding to =1>0, there exists positive integer m such that p,qm implies d(xp,xq)<. In particular, d(xp,xm)<1 for all pm. Let M=max{d(x1,xm),…,d(xm-1,xm),1}. For p,qm, d(xp,xq)<. For pm, q<m, d(xp,xq)d(xp,xm)+d(xm,xq)<1+M. If p,q<m, then d(xp,xq)M. Hence the proof.

**Theorem 3.5** Every convergent sequence in a MS (X,d) is Cauchy sequence in (X,d).

**Proof** Let lim(xn)=L in (X,d). Corresponding to >0, there exists positive integer m such that d(xn,L)<, for every nm. For positive integer p,qm, d(xp,xq) d(xp,L)+d(L,xq)<2.By the arbitrariness of >0, (xn) is Cauchy.

**Note** Converse may not hold in a general MS: Consider the sequence in the MS (0,1)(considered as a subspace of Ru). Let p,q be natural, p>q . d(=<⇐<⇐q>+1=m( ⇐p,q> m(. Thus the sequence is Cauchy bit it does not converge in (0,1).

=1+. Thus the rational sequence of partial sums (1,1+1/4,1+1/4+3/32,…) converges to , a non-rational number. In Ru, this sequence is convergent and hence Cauchy; so the sequence is Cauchy in the subspace of rational numbers but is not convergent there.

**Definition 3.3** Let x:N(X,d) be a sequence in the MS (X,d) and let k:NN be a strictly monotonically increasing function. Then x0k: N(X,d), (x0k)(n)=x(k(n)) (denoted by is called a subsequence of the sequence x. In particular, taking k(n)=n, every sequence is a subsequence of itself. Note that k(n)n for all natural n; hence lim(k(n))= as n tends to .

Example3.2If x(n)= and k(n)=2n, then the corresponding subsequence of x is is (=.

**Theorem 3.6** Let (xn) be a convergent sequence in a MS (X,d) such that lim(xn)=x. If is a subsequence of (xn), then lim()=x.

**Proof** For an arbitrary >0, there exists natural number m such that d(xn,x)< for nm. For nm,kn=k(n)nm , so that d(,x).Hence lim()=x.

**Note** If a subsequence of a sequence converges, then the sequence need not converge: consider the subsequence (2,2,2,…) of the sequence (2,-2,2,-2,…).

**Theorem 3.7** Let (xn) be a Cauchy sequence in a MS (X,d). Then (xn) is convergent iff (xn) has a convergent subsequence.

**Proof** If (xn) is convergent, every subsequence of (xn) is convergent. Conversely, let a CS (xn) in (X,d) has a convergent subsequence () converging to xX. Corresponding to >0, (taking maximum, if necessary) there exists positive integer m such that d(,x), d(xp,xq) for n,p,qm. Then nkmd(xn,x) d(,x)+d(xn,)<2 [ note that kn nkmHence lim(xn)=x.

**Definition 3.4(Complete MS )** A MS (X,d) is complete iff every CS in X is convergent in (X,d).

Example3.3The discrete MS is complete, since a CS in discrete MS is eventually constant.

Example3.3Ru is complete. Below we prove the chain: LUB axiomnested Interval Property(NIP)Bounded sequence has convergent subsequenceEvery Cauchy sequence is convergent(CSC).

**LUB axiomNested Interval Property(NIP)**

**NIP** Let {[an,bn]} be a sequence of closed intervals in R with the properties: (1) In+1In for all natural n, where In=[an,bn], (2) lim(bn-an)=0, then is a singleton.

**Proof** Sequence(an) is monotone increasing and bounded above by b1 and (bn) is monotone decreasing and bounded below by a1; hence by Monotone Convergence Theorem(which is derived from LUB property), lim(an)=sup{an: n natural}=L and lim(bn)=inf{an: n natural}=M exists. From definition of sup and inf, anL, bnM. Also L-M=Lim (an-bn)=0. Thus anLbn, for all n. If also anbn, for all n, then 0L/-anbn-an for all n implies L=lim(an)=L/. Hence is a singleton.

**NIP Every bounded sequence has a convergent subsequence**

**Proof** Let (an) be a bounded sequence of real numbers. Then there exists M>0 such that -ManM, for all n. One of the subintervals [-M,0] and [0,M] contains an for infinitely many values of n; call the subinterval I1. Repeating the process, we get a sequence(In) of closed intervals each of which contains an for infinitely many n and lim=0. By NIP, is a singleton, say,c. Choose from I1, from I2 with k2>k1(possible since I2 contains an for infinitely many n), and so on. Note that for all n; since lim=0, lim(=c.

**Every bounded sequence has a convergent subsequenceEvery CS is convergent(CSC)**

**Proof** Let (xn) be a CS. Then (xn) is bounded and hence , by assumption, has a convergent subsequence and, by Theorem above, is convergent.

**Note** In Ru, the notions of LUB-completeness, NIP-completeness and CSC are equivalent in the sense that any one can be derived from any other; we have proved a part of the equivalence above, the circle of implications can be completed. In a general MS, LUB-completeness and NIP-completeness cannot be formed meaningfully; CSC is used instead.

1. (0,1) is not complete.
2. The MS P[a,b] of all polynomials defined on [a,b] with uniform metric is not complete; indeed, in view of Weierstrass Approximation Theorem, uniform limit of a sequence of polynomials need not be a polynomial.
3. Q ,considered as a subspace of Ru, is not complete.

**Lemma 3.1**  Let (Y,dY) be a subspace of a MS(X,d), (yn) is a sequence in Y. Prove that lim(yn)=yY in (Y,dY) iff lim(yn)=y in (X,d).

**Proof** lim(yn)=yY in (Y,dY)⇔For everyopen set V in Y containing y, there exists m such that ynV for nm⇔ ynUY for nm, U open in X⇔ ynU for nm⇔ lim(yn)=y in (X,d).

**Theorem 3.8** Let (Y,dY) be a subspace of a MS(X,d). If Y is complete, then Y is closed.

**Proof** Let xY/-Y. Then there exists a sequence (yn) in Y such that lim(yn)=x in (X,d), ynx for all n. Thus (yn) is Cauchy in (X,d) and hence in (Y,dY). Since Y is complete, lim(yn)=yY in (Y,dY) and hence in (X,d). Since limit of a sequence in a MS is unique, x=y, contradiction.

**Note** Converse may not hold; consider the closed but incomplete subset (0,] of the MS (0,1] with usual metric; note that (0,1] is incomplete under usual metric. Converse holds if X is complete.

**Theorem 3.9** Let (X,d) be complete MS and Y be a subspace of X. If Y is closed, Y is complete.

**Proof** Let (yn) be a CS in Y, then (yn) is CS in X and ,X being complete,(y­n) converges in (X,d), say, to xX. Then x Y/Y, since Y is closed. Thus lim(yn)=x in (Y,dY).

**Corollary** Let (X,d) be a complete MS and YX. Y is complete iff Y is closed.

For the system R of real numbers, the LUB-completeness is equivalent to NIP which states that for every decreasing sequence of closed intervals ((an,bn)) with the length bn-an tending to 0 as n tends to , there exists exactly one point common to all these intervals. In an effort to generalize this result to MS, we replace closed intervals by closed sets and length of an interval by diameter of a closed set:

**Theorem 3.10** (Cantor’s Intersection Theorem) A MS (X,d) is complete iff for any sequence (Fn) of nonempty closed sets satisfying (1) Fn+1Fn for all n and (2) lim(d(Fn))=0 as , consists of exactly one point.

**Proof** (**Necessity)** Let (Fn)be a sequence of nonempty closed sets satisfying (1) Fn+1Fn for all n and (2) lim(d(Fn))=0 as . Choose xnFn (, for all n. The sequence (xn) is Cauchy ; since , for m>n,m,n natural, FmFn; so xm,xnFn, implying d(xm,xn)d(Fn)0 as n. Thus d(xm,xn)0 as n; hence (xn) is Cauchy. Since (X,d) is complete, there exists xX such that lim(xn)=x. We shall prove ={x}. For a fixed positive integer m, xnFm for all nm; hence x=lim(xn)=Fm, for all m; thus {x} . If y, then , since x,yFn, for all n, 0d(x,y)d(Fn) and lim(d(Fn))=0 as . Thus, d(x,y)=0 and hence x=y. Hence consists of exactly one point.

**(Sufficiency)** Let (xn) be a CS in (X,d). Let Fn={xn,xn+1,…}for all natural n. Obviously Fn+1Fn and hence for all n. Since (xn) is a CS, d()=d(Fn)0 as . By hypothesis, consists of exactly one point, say, xX. Since x,xn for all n, d(x,xn)d()=d(Fn)0 as n. Thus (xn) converges in (X,d).

**Note** None of the conditions that Fn’s are closed sets and d(Fn)0 as can be dropped as seen through following examples:

* Let Fn=[n,: lim(d(Fn))0: =
* Fn=: Fn’s are not closed in Ru:=.

**Note** If d and d1 are two equivalent metrics on a set X, the collections of Cauchy sequences in (X,d) and (X,d1) may be different; it may happen that (X,d) is complete while (X,d1) is not. N, the set of natural numbers, is complete w.r.t. usual metric whereas (N,d1) is incomplete where d1(x,y)=.

**BAIRE CATEGORY THEOREM**

A subset A of a MS (X,d) is dense iff B(x,r)A, for all xX and all r>0.

**Definition 3.5** A subset A of a MS (X,d) is No Where Dense (NWD) iff for every non-empty open set G of (X,d), there exists non-empty open set G\* such that G\*G and G\*A=.

Example3.4 Q is dense in R; N is NWD in R.

**Note** If AB and if B is NWD, then A is NWD. If A1,A2 are NWD, prove A1A2 is NWD.

* Let A1,A2 be NWD. Let U, U open. Since A1 is NWD, there exists non-empty open set U\* such that U\*U and U\*A1=. Since A2 is NWD, there exists non-empty open set V\* such that V\* U\* U and V\*A2=. Then V\*( A1A2)( V\*A2)( U\*A1) =. Thus A1A2 is NWD.

**Theorem 3.11** Let (X,d) be a MS and AX. A is NWD in (X,d) iff is dense in (X,d).

**Proof** Let A be NWD. Let G be any nonempty open set( in particular, may be a ball). By definition, there exists non-empty open set G\* such that G\*G and G\*A=. Since G\*A=, G\*=. Thus G\*X-. Hence G\*G(X-). Thus is dense in (X,d).

Conversely, let be dense in (X,d). Let G be any nonempty open set in (X,d). Then G(X-).Let G\*= G(X-). Then G\* is open and G\*G and G\* X-X-A; hence G\*A=. Thus A is NWD.

**Theorem 3.12** Let (X,d) be a MS and AX. A is NWD iff =.

**Proof** Let = and let G be any nonempty open set in (X,d). Since =, G is not a subset of ; hence G(X-). Let G\*= G(X-). Then G\* G and G\*A=. Thus A is NWD.

Conversely, let G be any nonempty open set in (X,d). Since A is NWD, there exists nonempty open set G\*such that G\* G and G\*A=. Thus AX- G\* which implies =X-G\*, that is, G\*X-. Then (X-)G⊇ G\*G= G\*, that is, G is not a subset of . Hence contains no nonempty open set and consequently, =.

**Corollary** If A is NWD, then is NWD, since int( = int(.

**Corollary** Let A be an open set in a MS (X,d). Then X-A is NWD iff A is dense in (X,d).

Let X-A be NWD. By Theorem above, Int(X-A) = Int()=(since A is open, X-A is closed). If G be any nonempty open set, G is not a subset of X-A. Thus GA, for every nonempty open set G; hence A is dense in (X,d).

Conversely, let X-A be not NWD. Then Int(). Since A is open, Int(X-A) . But A Int(X-A) =; hence A is not dense in (X,d).

**Definition3.6** A subset A of a MS (X,d) is (1) of first category iff A is expressible as a countable union of NWD sets; (2) of second category iff A is not of first category.

**Note** Every NWD set is of first category but not conversely e.g. Q in Ru.

**Theorem 3.13** In a MS (X,d), (1) any subset of a set of first category is of first category, (2) the union of countable collection of sets of first category is of first category, (3) any countable set of X is a set of first category if X contains no isolated point.

**Proof** (1) Let A be a set of first category, that is, A=, where each Ui is NWD set in X. If BA, then B=BA=B[]= is of first category since each , being a subset of NWD set Ui, is NWD in X, for each i.

(2)countable union of countable sets is countable.

(3) If X contains no isolated point, then every singleton is NWD and each countable set is countable union of singletons.

**Theorem 3.14** (Baire’s Category Theorem) A complete MS is of second category.

**Proof** If possible, let (X,d) be a complete MS and X=, where each Ui is of first category. Let G be any nonempty open set in X. Since U1 is NWD, there exists nonempty open subset G1\*G such that G1\*U1=. Take x1 G1\*. Since G1\* is open in X, there exists r1>0 such that B(x1,r1) G1\*. Let a1=min{. Let F1=B[x1,a1](closed ball). F1 is a nonempty closed set and F1= B[x1,a1]B(x1,r1) G1\*G. F1U1= since G1\*U1= and F1 G1\*. Also d(F1)=2a11.

Since U2 is NWD, corresponding to open set B(x1,a1), there exists open set G2\* such that G2\*B(x1,a1) and G2\*U2=. Take x2 G2\*. Since G2\* is open , there exists r2>0 such that B(x2,r2) G2\*. Let a2=min{. Let F2=B[x2,a2]. Then F2B(x2,r2) G2\*B(x1,a1) B[x2,a2]=F1. F2 has the properties: F2 non-empty closed set, F2U2=, d(F2)=2a2.

Suppose , as an induction step, that F1,F2,…,Fk have been constructed such that each Fi (i=1,…,k) is nonempty closed set, FiUi=, d(Fi). Now Uk+1 is NWD and B(xk,ak) is an open set where Fk=B[xk,ak]. Thus there exists Gk+1\* B(xk,ak) such that Uk+1 Gk+1\*=. Take xk+1 Gk+1\*. There exists rk+1>0 such that B(xk+1,rk+1) Gk+1\*. Take ak+1=min{}. Let Fk+1=B[xk+1,ak+1] B(xk+1,rk+1) Gk+1\* B(xk,ak) B[xk,ak]= Fk. Then Fk+1 is a nonempty closed set, Fk+1Uk+1=, d(Fk+1)=2ak+1.

By induction, we get a sequence {Fn} of sets satisfying the properties: (1) each Fn is nonempty closed set, (2) FnUn=, (3) d(Fn), (4)Fn+1Fn. Thus by Cantor’s Intersection Theorem, there exists x0X such that ={x0}. Now x0Fn implies x0∉Un for all n; hence x­0∉=X, contradiction. Thus (X,d) is of second category.

**Corollary** Let (X,dX) be a complete MS. Then (1)every closed subset Y of X is of second category in (Y,dY), (2) if C(X) is of first category in (X,d), then X-C is of second category in (X,d), (3) if X contains no isolated point, then X is uncountable.

**Proof** (1)Since Y is closed in the complete MS (X,d), (Y,dY) is complete MS. Hence Y is of second category in (Y,dY).

(2)If both C and X-C is of first category, then X is of first category in (X,d), contradiction.

(3)If X were countable, then, since X does not contain any isolated point, X could be expressed as countable union of singleton(which are NWD), contradicting (X,d) complete.

**Note** Let AX, (X,d) a MS. The property of A to be NWD, of first or of second category is not an intrinsic property of the subset A but very much dependent on the ambient space (X,d). For example, N is a closed subset of the complete MS (R,d) (real number with usual metric); hence (N,dN) is complete MS and hence is of second category but N is **not** of second category in (R,d): N={n}, each {n} is NWD sets in (R,d).

We shall now consider an alternative proof of Baire’s Category Theorem. In the process it will be shown that in a complete MS every open subset is of second category.

**Definition3.7** A MS (X,d) is a Baire space iff the intersection of each countable family of open dense sets in X is dense in X.

**Theorem 3.15**  A complete MS is a Baire space.

**Proof** Let {Dn} be a countable collection of open dense subsets in a complete MS (X,d). Let U be any nonempty set in (X,d). To prove: U.

Since D1 is dense and open in X and U is a nonempty open set in X, UD1) is open in X. Thus there exists xX and r>0 such that B(x,r) UD1.Let a1=min{ }.ThenB1=B(x,a1) satisfies the properties B1 B(x,r) UD1, d()=d(B1)=2a1<1. We now consider B1 and D2 to find an open ball B2 such that B1D2 where d()<1/2. By induction, we can find a sequence {Bn} of nonempty open balls such that Bn-1Dn where d()<1/n for each n. By construction, . Since is a sequence of nonempty closed descending sets with d(0 as n, by Cantor’s Intersection Theorem,. Hence the proof.

**Theorem 3.16**  Let (X,d) be a MS which is Baire space. If X=, then int( ) for at least one n. Thus X cannot be of first category.

**Proof** X= =. Thus =X-= and is open in (X,d) for each n. Since X is a Baire space, this shows that not every is dense . Hence X for at least one n, that is, int(=X- for at least one n, that is, An is not NWD for at least one n. Thus X is of second category.

Taking into account above two theorems, we have

**Corollary** Every complete MS is of second category. The converse may not hold:

**Note** Let X stand for the open interval (a,b) endowed with usual metric. (a,b)is clearly dense in [a,b](under usual metric). Let G1,G2,…,Gn,… be a sequence of dense open sets in X. Then Gi=Hi(a,b), where each Hi is dense and open in [a,b]. Now ,(a,b),H1,H2,… is a sequence of dense open sets in the Baire space [a,b] and hence (a,b) is dense in [a,b] and ,therefore, in (a,b).Hence (a,b) is a Baire space and hence is of second category in itself though the space (a,b) is not complete.

**A Few Consequences**

1. In a Baire space (and hence in a complete MS) (X,d), a set of first category has void interior.( **Note** Closure of a set of first category may have nonvoid interior: =R has nonvoid interior in R though Q has void interior in R)

Proof: Let B be a first category set in a Baire space (X,d). Then B=, where each Bn is NWD in X. Let U be any open set in X such that UB. Now U implies X- X-U. Since each Bn is NWD, Int = and hence is open and dense set in X, for each n. Since X is Baire, is dense in X. Thus X-U is dense in X, that is, X-U==X and hence U=.

1. The complement of every set of first category in a complete MS is dense in X
2. Every nonempty open set in a complete MS is a set of second category.

Example 3.5 Use Baire Category Theorem to prove that R is uncountable.

**Proof** If possible, let R be countable, say, R={x1,..,xn,..}. Since each singleton {xi} is NWD, R is of first category. But R, being complete MS, is of second category, contradiction. Hence R ids uncountable.

**CHAPTER IV**

**Compactness in Metric Spaces**

Many of the important theorems of classical real analysis (like attaining of bounds of continuous functions defined on closed bounded inervals)hold for closed and bounded intervals and fail for other types of intervals. These desirable properties possessed by closed, bounded intervals, which are not shared by other type of intervals, when generalized to arbitrary MS gives rise to the definition of compactness.

**Definition 4.1** Let X be a nonempty set and AX. A collection U of subsets of X is called a cover or covering of A iff AU}. A subcollection U0 of U is a subcover of U for A iff AU0} holds. If A=X, we say U0 is a subcover of U. If U0 contains finite (countable) number of elements, then U0 is called a finite (countable resp.) subcover of the cover U.

**Definition 4.2** Let (X,d) be a MS. A cover U of X is an open cover of X iff every element of U is an open set of (X,d).

**Definition 4.3** A MS (X,d) is compact iff every open cover U of X has a finite subcover U0. A subset A of (X,d) is compact iff (A,dA) is compact.

**Theorem 4.1** AX, (X,d) is a MS. A is compact iff every cover of A by open sets(not necessarily contained in A) in (X,d) has finite subcover.

**Proof** Let U={} be an open cover of the compact subset A of the MS (X,d) in (X,d). Since AU}, A=U}. Since U\*={U} is an open cover of A in (A,dA), U\* has a finite subcover { for A; thus A==. Hence A; thus every open cover of A by sets open in (X,d) has a finite subcover .

Conversely, let every open cover of A by sets open in (X,d) has a finite subcover: to prove (A,dA) is compact. Let V={} be an open cover of A in (A,dA). Since, for each , is open in (A,dA), there exists open in (X,d) such that A, for each . U={} is an open cover of A in (X,d); hence there exists finite subcover { of U such that A. Then A==, proving that (A,dA) is compact.

**Note** If d1 and d2 be topologically equivalent metric on a set X (that is, if the collection of open sets in (X,d1) and (X,d2) are coincident), then (X,d1) is compact iff (X,d2) is compact.

(2)Every finite subset of a MS is compact and hence, in particular, every finite MS is compact.

(3) No infinite subset A of an infinite discrete MS X is compact: the open cover {{a}:aA} of A in X has no finite subcover.

(4) Ru is not compact; The open cover {(-n,n):nN} of R has no finite subcover. (0,1) is not compact since the open cover has no finite subcover: if is a finite subcover and if m>max{n1,…,nk}>0, then (0,1)-, contradiction.

(5) Let (xn) be a sequence in a MS (X,d) converging to xX. Then A={xn}{x} is compact subset in (X,d): If {}be an open cover of A in (X,d), then there exists in {} such that x; since is open, there exists natural k such that xi for i>k; also there exists in {} such that xi, i=1,…,k. hence {,,…,} is a finite subcover of {}.

**Theorem 4.2** A compact subset A of a MS (X,d) is closed and bounded.

**Proof** If possible, suppose A is not closed. Then a limit point x0 of A does not belong to A. Define Gn=X-B[x0,] (n natural). Then {Gn} is an open cover of A: =X-{x0}⊇A. Since A is compact, {Gn} has a finite subcover . Let n0= max{n1,…,nk}. Then =⊇A. Thus B[x0,]A= and hence B(x0,A= contradicting x0 is a limit point of A. Thus A is closed in (X,d).

Obviously, U= {B(x0,n): n natural} is an open cover of X. Since X is compact, U has a finite subcover, say, {B(x0,n1),…, B(x0,nk)}. Let ni=max{n1,…,nk}. Then X=B(x0,ni). Thus d(X) 2ni and hence X is bounded.

**Note** Closed and bounded subset need not be compact in a general MS: Consider S=[0,1] with discrete topology. A={N} is closed and bounded in S but the open cover U={} has no finite subcover; thus A is not compact. However, the converse holds in Rn, for every natural n.

**Theorem 4.3** (Heine-Borel Theorem for Rn) Every closed and bounded subset of Rn is compact.

**Note** The closed and bounded intervals in R are compact whereas other type of intervals are not.

**Theorem 4.4**  Every closed subset C of a compact MS (X,d) is compact.

**Proof** Let U={} be an open cover of the closed subset C of the MS (X,d) in (X,d). Then U { X-C} is an open cover of the compact MS X; let {} be the corresponding finite subcover: CX=. Thus C(since C(X-C)=), proving C is compact.

Example 4.1 [1,5,6,7,8} is closed subset of the compact space [1,8] and hence compact.

**Definition 4.4** (Finite Intersection Property)A family U of sets in a MS X is said to have Finite Intersection Property (F.I.P.) iff the intersection of the members of each finite subfamily of U is nonempty.

**Theorem 4.5** A MS (X,d) is compact iff for every collection of closed sets {} in X having f.i.p., .

**Proof** Let (X,d) be compact and {} be a collection of closed sets in (X,d) having f.i.p. If possible, let . Then X=X-; hence is an open cover of X and let { be the corresponding finite subcover of X. Thus X= implying =, contradicting f.i.p.

Conversely, let for every collection of closed sets {} in X having f.i.p., . Let U={} be an open cover of X. Then X=. Thus =; conclusion is that the family of sets does not possess f.i.p.; hence there exists a finite subcollection {}such that =X-; hence X= providing finite subcover of U.

**Total Boundedness, Sequential Compactness and Bolzano-Weierstrass Compactness**

In the setting of a MS, there are certain equivalent descriptions of compactness in terms of convergence of sequences or limit points of sets. Below we make some such deliberations. We shall see that the conclusion of Bolzano-Weierstrass Theorem on R (every bounded infinite subset of R has a limit point in R) may not hold good for a general MS X, but a condition stronger than the B.W. property is equivalent to compactness of X. Since B.W. property of R is known to be equivalent to its completeness property , it turns out that compactness of a MS has something to do with the completeness property of the space.

Following concept, which is strictly stronger than boundedness, helps us to frame an analogue of B.W. theorem in an arbitrary MS.

**Definition4.5** Let (X,d) be a MS and let YX.Let >0. A finite subset A={a1,…,an} of Y is an -net in Y iff for each y in Y, there exists ai (1in)in A such that d(y, ai)<. In particular, we may consider the concept of an -net in (X,d).

**Lemma 4.1** A MS (X,d) is bounded iff it has an -net for some >0.

**Proof** If X is bounded, then X=B(a,s) for some aX and some s>0. Then {a} is an s-net for X.

Conversely, let H={a1,…,an} be an -net for X. Let x,yX. Then there exists ai,ajH such that d(x,ai)< and d(y,aj)<. Then d(x,y)<2+ M, where M=max{d(ai,aj): ai,ajH}. Hence d(X) 2+ M, proving X is bounded.

**Definition4.6** A subset Y in a MS (X,d) is totally bounded iff for every >0, there exists an -net in Y. In particular, we may take Y to be X to get definition of total boundedness of (X,d).

**Theorem 4.6** Let (X,d) be a MS, A is TB, BA. Then B is TB.

**Proof**  Let >0 be given . By total boundedness of A, there exists an /2- net, say, H={a1,…,an} in A. For each bB, there exists ak(1kn) such that d(ak,b)</2. Let C={}H such that for each C, there exists bB such that d(,b) </2. For each , choose one bkB such that d(, bk) </2. Consider S={b1,…,br}B. We claim that S is an -net for B: For any bB, there exists C such that d(,b) </2. Now d(b,bk)d(b,)+d(,bk)<.

**Theorem 4.7**  Let (X,d) be a MS and YX. Y compact⇒ Y totally bounded⇒ Y bounded.

**Proof** Let Y be compact and let >0 be given . The open cover {B(y,):yY} is an open cover of Y; let {B(y1,),…,B(yn,)} be the corresponding subcover. Since Y, ={y1,…,yn} is an -net for Y. By arbitrariness of >0, Y is totally bounded.

Next, let Y be totally bounded. Then Y has an 1-net, say, A={a1,..,an}. Let y,zY. Then there exists ai,ajA such that d(y,ai)<1 and d(z,aj)<1. Then d(y,z)d(y,ai)+d(ai,aj)+d(aj,z)<2+max{d(as,at): 1s,tn}=M(say), which is independent of y,zY. Thus d(Y)M and hence Y is bounded in (X,d).

**Note** None of the implications in the above theorem can be reversed in general:

**Boundedness may not imply total boundedness**

Example 4.2 The MS [0,1] with discrete metric is bounded but not totally bounded: it has, say, no ½-net.

As a second example ,consider the MS (l2,d), where l2 is the set of all real sequences (xn) such that < and d((xn),(yn))=. Consider the subset A={**x**=(xn)l2:d(x,**0**)=1}, where **0**=(0,0,…). Clearly A is bounded: d(**x**,**y**)d(**x**,**0**)+d(**y**,**0**)=2 for all **x**,**y**A.

We next prove that A is not totally bounded by showing that A has no -net. Consider the subset B={**e1**,**e2,**…} of A, where **ei** is the sequence whose i th term is zero and all other terms are zero. Then , for mn, d(**em**,**en**)=. Hence any open ball of radius can contain at most one point of B. Since B is infinite, it is not possible to cover A by finitely many balls of radius .

A is closed; let ycl(A). Let (xn) be a sequence in A converging in l2 to y. Thus for any r>0, there exists natural m such that d(xn,y)<r, for every n>m-1. Thus d(xm,y)<r. So d(y,0)d(y,xm)+d(xm,0)<1+r, for every r>0. Hence d(y,0)1. Again 1=d(xm,0)d(xm,y)+d(y,0)<r+d(y,0), for all r>0. Thus 1d(y,0). Hence d(y,0)=1 and yA.

**Note** Above example also shows that the Heine-Borel Theorem does not hold for a general MS. A is closed and bounded but not compact since it is not totally bounded.

**In Ru, boundedness and total boundedness are equivalent concepts**

**Proof** Let A be a bounded subset of R. Then there exists a>0 such that A-a,a]. For a given >0, choose natural n such that and then [-a,-a+, [-a+, -a+],…,[-a+] clearly cover A: thus {-a,-a+,…, -a+,} is an -net for [-a,a]. By arbitrariness of >0, [-a,a] and hence the subset A, is totally bounded.

**Total Boundedness may not imply compactness.**

**Proof** Thus (1,2) is bounded and hence totally bounded in Ru but is not closed and hence is not compact.

**Completeness and Total boundedness are mutually independent concepts**

**Proof** R is complete but not bounded and hence not totally bounded.

**A bounded complete MS need not be totally bounded**

**Proof** MS [0,1] with discrete metric.

**Note** f: (0,1][1,), f(x)=1/x is a homeomorphism (MSs considered as subspaces of Ru). (0,1] is bounded and totally bounded whereas [1,) is unbounded and hence not totally bounded. Also (0,1] is not complete whereas [1,), being a closed subspace of the complete MS Ru, is complete. Thus none of boundedness, total boundedness and completeness are topological properties.

One of the most fundamental results for the real line R is the Bolzano-Weierstrass Theorem which is an equivalent version of completeness property of R( considered as an Archimedean ordered field) and which states that every bounded infinite subset of R has a limit point in R. Stated in terms of sequence, it states that every bounded sequence in R has a convergent (and hence Cauchy)subsequence. But these results are false in a general MS. If we consider the set R with discrete metric d, then every subset and all sequences are bounded in (R,d). But in (R,d), no subset can have a limit point and there does not exist any convergent sequence (other than the eventually constant sequence). In the next theorem, an analogue of BW Theorem and its converse are established which additionally gives a characterization of total boundedness. To prove the theorem, we need the following lemma:

**Lemma 4.2** Let (X,d) be a TBMS and Abe an infinite subset of X. Corresponding to arbitrary >0, there exists infinite subset B of A such that d(B)<.

**Proof** Since X is TB, there exists an /3 net in X, say, {x1,…,xn}. Thus X=. Hence A=AX=A]. Since A is infinite, at least one of )A (i=1,…,n) must be infinite, we denote any such subset as B. Then B is an infinite subset of A and d(B)d[)]=/3<.

**Theorem 4.8**  A MS (X,d) is totally bounded iff every sequence in X has a Cauchy subsequence.

**Proof** Let (X,d) be TBMS. Let (x­n) be a sequence in X. If {xn} is finite, then there is at least one term of the sequence appearing for infinitely many values of n; thus, in this case, the sequence has a subsequence of constant term which is therefore Cauchy.

Next, let {xn} be infinite. By Lemma above, there is an infinite subset B1 of {xn} such that d(B1)<1. Proceeding inductively, there exists an infinite subset Bk+1 of Bk such that d(Bk+1)< . Choose B1, B2 with n2>n1(possible since B2 contains infinite number of terms of the sequence (xn)) and , in general, Bk+1 with nk+1>nk . We want to prove () is a Cauchy sequence in (X,d). Let >0 be given. There exists natural m such that 1/m<. For natural km, BkBm. Hence for p,qm, d(,)d(Bm)<<. Thus () is a Cauchy subsequence of the sequence (xn).

Conversely, let (X,d) be not TB. Then there exist 0>0 such that there is no 0-net in X. Let x1X. Since {x1} is not an 0-net in X, there exists x2X such that d(x1,x2)0. Since {x1,x2} is not an 0-net in X, there exists x3X such that d(x2,x3)0 and d(x1,x3)0. Proceeding in this manner, we get a sequence (xn) in (X,d) such that distance between any two terms of the sequence is 0. The sequence (xn) has no Cauchy subsequence. Thus the converse part is proved.

Since in the real line R, a CS converges and a bounded set is totally bounded, the first part of above theorem, when interpreted for the real line, proves Bolzano-Weierstrass Theorem. The failure of BW Theorem to hold in a general MS motivates the following definition:

**Definition 4.7** A MS (X,d) is sequentially compact iff every sequence in X has a convergent subsequence.

**Theorem 4.9**  For a MS (X,d), the following are equivalent:

1. X is complete and totally bounded, (2) X is sequentially compact, (3) X is BW Compact,(4) X is compact.

**Proof** (1)⇒(2) Let X be complete and totally bounded and let (xn) be a sequence in (X,d). Since X is totally bounded, (xn) has a Cauchy subsequence, say, (. Since X is complete, () converges to x (say) in X. Hence (X,d) is sequentially compact.

(2) ⇒(1) Since X is sequentially compact, every sequence in X has a convergent subsequence which is , naturally, a Cauchy subsequence. By Theorem above, X is totally bounded. Let (xn) be a CS in (X,d). Since X is sequentially compact, (xn) has a convergent subsequence (. Thus (xn) has a CS which has a convergent subsequence and hence (xn) must converge. Thus X is complete.

(2) ⇒(3) Let A be an infinite subset of a sequentially compact MS (X,d). We can extract a sequence (xn) of distinct terms from A. Since X is sequentially compact, (xn) has a convergent subsequence () converging to x(say)X. Then every neighbourhood of x contains the subsequence eventually, that is, intersects A at infinitely many points. Thus x is a limit point of A.

(3) ⇒(2) Let (xn) be a sequence in X. If the range {xn} is finite subset, then the value of a term of the sequence must be repeated infinitely often. Then (xn) has a constant subsequence which is obviously convergent. If {xn} is infinite set, then {xn} has a limit point x (say) in X. B(x,1)A is an infinite set: choose B(x,1)A. B(x,1/2)A is an infinite set; choose B(x,1/2)A with n2>n1. Proceeding in this manner, we get a subsequence () of (xn) with d(,x)<Thus () is a convergent subsequence of (xn).

(4) ⇒(3) Let A be an infinite subset of X and , if possible, let A have no limit point in X. Then for all xA, there exists rx>0 such that B(x,rx)A={x}. Since A has no limit point in X, A is closed in X. Since X is compact, the open covering {B(x,rx):xA}(X-A) has a finite subcovering: AX=(X-A). Then A, contradiction since the set at the left is infinite while the set at right is finite.

To complete the cycle of implications, we next show that every sequentially compact MS is compact. For that we require the following definition:

**Definition 4.8** Let U= be an open cover of a MS (X,d). A real number >0 (if it exists) is a Lebesgue number corresponding to the open cover U iff every subset A of X with d(A)< is contained in for at least one .

**Note** There are open covers of a MS without Lebesgue number.Consider the open cover {} of the MS (0,1) with usual metric. If possible, let >0 be a Lebesgue number for the open cover. Observe that A=(0,/2)(0,1) with d((0,/2))=/2< but A is not contained in any single member of the open cover. Hence the open cover has no Lebesgue number.

**Lemma 4.3**  Every open cover of a sequentially compact MS (X,d) has a Lebesgue number.

**Proof** If the assertion is false, then there exists an open cover U= of X having no Lebesgue number. Then , for each natural n, since 1/n is not a Lebesgue number, there exists subset An of X such that d(An)<1/n and An is not contained in any single U

Choose xnAn, nN.(xn) has a convergent subsequence ( converging, say , to x0 in X. Since U is a cover of X, x0, for some in U .Since is open, there exists r>0 such that B(x0,r). Since converges to x0, corresponding to r>0, there exists mN such tht km implies d(,x0)<r/2. Since lim d(A­n)=0, there exists pN such that kp implies d(<<.Let q= max{m,p}. Let y. Since and since d(<<, so d(,y) <. Also d(,x0)< (since qm). Thus d(y,x0)<r. So yB(x0,r). So B(x0,r), contradiction. Hence the result.

**Theorem 4.10**  Every sequentially compact MS (X,d) is compact.

**Proof** Let U= be an open cover of X. By the lemma, U has a Lebesgue number . Since (X,d) is sequentially compact, it is totally bounded. Thus X has a /3 –net, say, H={x1,…,xn}. Thus X=. Let xX. There exists i, 1, xB(xi,). Now d[B(xi,)]<. By definition of Lebesgue number, there exists such that B(xi,), i=1,…,n. Thus X=; hence {} is a finite subcover of U.

**Theorem 4.11**  If (X,d) is compact, (X,d) is complete.

**Proof** Let (xn) be a CS in (X,d). Since X is compact, X is sequentially compact; thus (xn) has a convergent subsequence (), convergent to x0 in X. Thus (xn) is convergent. Hence (X,d) is complete.

Completeness may not imply compactness: consider the set [0,1] with discrete metric. As proved above, Completeness together with TB imply compactness.

**The special case of Ru**

**Theorem 4.12** (Bolzano-Weierstrass) Every bounded infinite subset A of R has a limit point in R.

**Proof** Since A is bounded, A[a,b], there exists real numbers a,b. Let X=[a,b].(X,d) is a metric subspace of Ru. We first prove that (X,d)is compact. Let (xn) be a CS in (X,d). Then (xn) is a CS in Ru. Since Ru is complete, (xn)x0 in Ru. Now axnb for all n. Thus , by squeeze theorem, ax0b. Thus, x0X. Thus (X,d) is a complete MS. Given >0; choose natural number m such that . Divide [a,b]into m subintervals of equal length h so that h=. For any x[a,b], x[xr-1,xr], r{1,2,…,n}. Then d(x,xr)h<. Thus (X,d) is TB and complete and hence compact. Since A is an infinite subset in the compact and hence BW Compact space (X,d), A has a limit point in X and hence in R.

**Theorem 4.13**(Heine-Borel) Every closed and bounded subset X of R is compact.

**Proof** Since X is bounded, there exists real numbers a,b such that X[a,b]. X is closed in Ru and hence closed in (A,d), where A=[a,b] (since X=XA). Since A is compact, X is compact.

**Theorem 4.14** Bounded sequence of real numbers has a convergent subsequence.

**Proof** Let (xn) be a bounded sequence of real numbers. Then there exist real numbers a,b such that axnb, for all n. A=[a,b] is compact and hence sequentially compact MS. Thus the sequence (xn) has a convergent subsequence in (A,d). Since converges in (A,d), it converges in Ru.

**Theorem 4.15**  Set of all subsequential limits of a sequence of real numbers form a closed set.

**Proof** Let C be the set of subsequential limits of the sequence (xn) of real numbers. Let x be a limit point of C. Then for every >0, B(x,[C-{x}]. Let y B(x,[C-{x}]. There exists >0 such that B(y,) B(x, (since B(x,is an open set). B(y,) contains xn forinfinitely many values of n (since yA). All these xn for infinitely many n belong to B(x,, proving that x. Hence C is closed.

**CHAPTER V**

**Connectedness in Metric Spaces**

**Definition5.1** Let (X,d) be a MS and A,BX. A and B are separated iff B=A (closures in (X,d)).

Example 5.1 In Ru, (0,1) and (1,2) are separated; on the other hand, (0,1) and [1,2), though disjoint, are not separated. Any two disjoint sets in a discrete MS are separated.

**Theorem 5.1** Let (X,d) be a MS and A,B, then A and B are separated.

**Proof** Let d(A,B)=m. By definition of infimum, d(a,b)m, for all a and bB. If a, then a since B(a,m/2)B=. Thus A. Similarly B=.

**Note** In Ru, A=(-,0) and B=(0, are separated though d(A,B)=0.

**Theorem 5.2** Let (X,d) be a MS and A,B are disjoint subsets of X. If either both A and B are closed or both A and B are open, then A and B are separated.

**Proof** If A and B are both closed disjoint subsets of X, then B=AB=. Similarly A.

Next ,let A and B are disjoint open subsets of (X,d). If possible, let A and B are not separated. Then either B or A. suppose A. Let xA. Thus there exists r>0 such that B(x,r)A and B(x,r)[B-{x}]. Consequently, AB, contradiction.

**Theorem 5.3** Let (X,d) be a MS, GX, G=AB, where A and B are separated sets. (1) If G is open, then A and B are open, (2) If G is closed, then A and B are closed.

**Proof** If either A or B is , nothing remains to prove. Let A and B be non-empty.

(1)Let G be open. Let xAG. There exists r>0 such that B(x,r)G. Since xA, x∉. Hence there exists r1>0, r1<r such that B(x,r1)B=. Hence B(x,r1)A. Thus xA0. By arbitrariness of xA, A is open. Similarly B can be proved to be open.

(2) Let G=AB be closed. . Similarly B is closed.

**Disconnected and connected sets**

**Definition 5.2** A MS (X,d) is disconnected iff X can be expressed as union of two nonempty separated subsets. X is connected iff it is not disconnected.

Example 5.2In a discrete MS (X,d), any set A with at least two points is disconnected: {x} and A-{x} is a separation for A in (X,d).

Example 5.3 Q, with subspace topology, is not connected: A={x: x rational, x< and B={x: x rational,x> is a separation for Q.

Example 5.4 In R2, the set {(x,y):y=sin, 0<x{(0,y): -1} is connected(contrary to intuition)

Four equivalent characterizations of disconnected (hence connected) MS are as follows:

**Theorem 5.4** Let (X,d) be a MS. The following statements are equivalent:

1. (X,d) is disconnected
2. X is expressible as union of two disjoint nonempty open sets
3. X is expressible as union of two disjoint nonempty closed sets
4. There exists a non-empty proper subset of X which is both open and closed.

**Proof** (1)⇔(2)

Let (X,d) be disconnected. Then there exists nonempty separated subsets A,B such that X=AB. Since X is open in (X,d),by theorem above, A,B are open in (X,d). Conversely, if X= AB, A,B are disjoint nonempty open sets, then by some theorem above, A,B are separated in (X,d) and form a separation of X.

(1)⇔(3) Similar proof.

(3)⇔(4)

If X=CD, where C and D are disjoint nonempty closed sets in X, then C=X-D is open in X; since D, C is a proper clopen non-empty subset of (X,d). Conversely, if C be a proper clopen non-empty subset of (X,d), then X=C(X-C), C and X-C are separated since they are disjoint nonempty closed subsets of (X,d).

**Definition 5.3** Let (X,d) be a MS and YX. Y is disconnected iff (Y,dY) is disconnected, that is, iff there exist two nonempty subsets A and B of Y such that Y=AB, ClY(A)B=AClY(B)=. Y is connected iff it is not disconnected.

**Theorem 5.5** Let (X,d) be a MS and YX. Y is disconnected iff there exist nonempty subsets A,B in X such that (1)Y=AB and (2) ClX(A)B=AClX(B)= hold.

**Proof** Let Y be disconnected. Then there exist nonempty subsets A,B in Y such that Y= AB and ClY(A)B=AClY(B)=. Since ClY(A)= ClX(A)Y, it follows that ClX(A)B= ClX(A)(YB) (Since BY)= ClY(A)B=. Similarly AClX(B)=.

Conversely let there exist nonempty subsets A,B in X such that (1)Y=AB and (2) ClX(A)B=AClX(B)= hold. Now AClX(B)= implies Y[ AClX(B)]=, that is, AClY(B)=. Similarly other part

**Theorem 5.6** Let (X,d) be a MS and YX be connected. If YAB where A and B are separated in X, then either YA or YB.

**Proof** Since ClXAB=AClXB=, ClX (YAB)=(YA)ClX (YB)= ( Since YAA, Y). Thus YA and YB are separated sets in X. Moreover, Y=YX=Y(AB)=( YA)( YB).Thus YA and YB form a separation for the connected subset Y, contradiction, unless one of YA and YB is empth, that is, either YA or YB.

**Theorem 5.7** Let (X,d) be a MS and let {be a family of connected sets in X such that . Then is connected.

**Proof** Let be not connected. Then there exist non-empty separated sets P,Q in X such that =PQ. Since P and Q are non-empty, there exist and such that P and . Then and are separated sets and as such =, contradicting . Hence is connected.

**Connectedness in Ru**

**Theorem 5.8**  A subset E of Ru is connected iff x , yE, x<z<y implies zE.

**Proof** Let x , yE, x<z<y and z∉E. Let Az=E(-,z) and Bz=E(z,). Since z∉E, E=AzBz. (-,z] (-,z] and hence Bz=. Similarly, Az=. Thus Az,Bz form a separation of E; hence E is disconnected. Hence the necessity of the condition.

Conversely, let E be disconnected and let F,G be nonempty disjoint closed sets in E such that E=FG. Let x , yE, x<z<y implies zE. Choose xF and yG. Since FG=, xy. Suppose ,without loss of generality, x<y. Since x,yE, by assumption [x,y]={zE:x}E. Let s=sup(F[x,y]). Clearly the supremum exists(since F[x,y] is bounded above by y) and x, so that sE. we show that sF. Let >0 be arbitrary. There exists z F[x,y] such that s-<z<s+. Thus [(s-,s+E]F. Since (s-,s+E is an arbitrary open ball in the subspace E centred at s, it follows that sClEF=F (since F is closed in E). Then s<y (sy,sF,yG, FG=). As no point of [x,y] strictly greater than s can be in F (by definition of s), (s,y]G and hence [s,y]=[s,y]E==ClEG=G. Thus sG. Consequently, sGF=, contradiction.

**Corollary**

1. Ru is connected and a subset A of Ru with at least two distinct points is connected iff A is an interval.
2. No subset A of Ru, other than R and , can be clopen.

**CHAPTER VI**

**LIMIT AND CONTINUITY**

**(of a function from a MS to a MS)**

**Definition 6.1** Let (X,dX) and (Y,dY) be MSs, EX, f:E. Let p be a limit point of E. Limit of f as x tends to p is q, written as , iff for every >0, there exists >0 such that xE,0<dX(x,p)< implies dY(f(x),q)<.

**Note** p need be a limit point of E for the definition of limit to be meaningful.

Example 6.1 Let f:l2Ru , f((xn))=x1. Let a=(an) be a fixed element of l2. We shall prove . Since d2(x,a)==, corresponding to given >0, choosing =, d2(x,a)< implies .

**Theorem 6.1** (sequential criteria for existence of limit) iff for every sequence (pn)in E satisfying pnp for all n and lim(pn)=p(in (X,d)), lim(f(pn))=q holds in (Y,dY).

**Proof** Let . Let >0 be given. Corresponding to chosen , there exists >0 such that xE,0<dX(x,p)< implies dY(f(x),q)<. Let lim(xn)=p in (X,d), xnp for all n. Corresponding to >0, there exists natural number m such that nm implies dX(xn,p)<. Hence nm implies dY(f(xn),q)< proving the necessity part.

Conversely, suppose . Then there exists 0>0 such that for every >0, there exists x in E, xp, such that dX(x,p)< holds but dY(f(x),q)0 holds. Thus for 0>0 and for =1/n>0 (n natural), there exists xn in E such that 0<dX(xn,p)<1/n but dY(f(xn),q) 0 holds. Thus (xn) is a sequence in E satisfying xnp for all n and lim(pn)=p(in (X,d)) but lim(f(pn))q.

**Theorem 6.2** 1 and 2 implies q1=q2.

**Proof** Corresponding to >0, there exist 1>0, 2>0 such that xE,0<dX(x,p)<1implies dY(f(x),q1)< and xE,0<dX(x,p)<2implies dY(f(x),q2)<. Let = min{1,2}>0. Then xE,0<dX(x,p)< implies 0 dY(q1,q2) dY(f(x),q1)+ dY(f(x),q2)<2, proving dY(q1,q2)=0, that is, q1=q2.

**The special case of real valued function**

All results concerning limits of functions f : (X,dX)(Y,dY) holds , in particular, for g: (X,dX)Ru. But some results valid for functions g: (X,dX)Ru do not have their counterparts in the general case.

**Theorem 6.3 (1)** .

(2)Let f,g,h: D[]Ru, f(x)g(x)h(x) for all x in a deleted nbd. of cD and =L, then =L.

**Theorem 6.4** (Cauchy criteria)Let f:DRu, c is a limit point of D. exists iff for every real number >0, there exists >0 such that x,yD imply .

**Continuous function from a MS to a MS**

**Definition 6.2** Let (X,dX) and (Y,dY) be MSs, EX, f:E, pE. f is continuous at p iff for every >0, there exists >0 such that xE, dX(x,p)< implies dY(f(x),f(p))<, that is, B(p, )E f-1[B(f(x),f(p)). If f is continuous at every pE, then f is continuous on E.

**Note**

1. If p is an isolated point of E, then any function f:E is continuous at p. Let >0 be given. Since p is an isolated point, there exists >0 such that B(p,E={p}. Obviously, {p}= B(p,E f-1[B(f(x),f(p)), for any f:E. Keeping this in view, we discuss continuity of a function only at limit points of the domain of definition of the function.
2. Every constant function from a metric space to a MS is continuous.
3. Every function from a discrete MS to any MS is continuous.

**Theorem 6.5** (sequential criteria for continuity) Let (X,dX) and (Y,dY) be MSs, EX, f:E, pE. f is continuous at p iff for each sequence (xn) in E converging to p in (X,dX), the sequence (f(xn)) in Y converges in (Y,dY).

**Note**

1. For a function f continuous at p, (f(xn)) converges to f(p) need not necessarily imply (xn) converges to p: let f:RuRu , f(x)=x2, xn=(-1)n, n natural. Then (f(xn)) converges to 1 but (xn) does not converge.
2. Convergent sequences are preserved under continuous functions; however it is not necessarily true for Cauchy sequences: f : (0,)(0,), f(x)=1/x, sends Cauchy sequence (1/n) to non-Cauchy sequence (n).

Example 6.2 Define f:C[a,b]Ru by f(x)=x(t0), where t0 is a fixed number in [a,b]. Prove that f is continuous on C[a,b]( C[a,b] is with sup metric). Is f continuous if C[a,b] is endowed with the metric d(x,y)=.

**Proof** For x,yC[a,b],. Thus, for given >0, taking =, d(x,y)< imply <.

**Theorem 6.6** (some equivalent formulation of continuity) Let (X,dX) and (Y,dY) be MSs and f: (X,dX) (Y,dY). Following are equivalent:

1. f is continuous on X
2. V is open in Y implies f-1(V) is open in X
3. V is closed in Y implies f-1(V) is closed in X
4. f(, for any AX.

**Proof**

(1)⇒(2) Let xf-1(V). Then f(x)V and V is open. Thus there exists >0 such that B(f(x),)V. Since f is continuous at x, corresponding to>0 , there exists >0 such that f(B(x,) B(f(x),)V, that is, B(x,)f-1(V). Thus x is an interior point of f-1(V). By arbitrariness of xf-1(V), f-1(V) is open .

(2) ⇒(3) Let V be closed in Y; then Y-V is open in Y. By assumption, f-1(Y-V)=f-1(Y)-f-1(V)=X-f-1(V) is open in X and hence f-1(V) is closed in X.

(3) ⇒(4) f-1( is closed in X and Af-1(f(A)) f-1(; hence f-1(), proving the assertion.

(4) ⇒(1) Let lim(xn)=x but lim(f(xn))f(x). Then there exists r>0 such that f(xn)B(f(x),r) for infinitely many n; let A ={xn: f(xn)B(f(x),r)}. By assumption, f(. Let ()be a subsequence of (xn) extracted from A; clearly lim()=x; thus x though f(x), contradiction.

**Theorem 6.7**  Let f:XY and g:YZ be functions where X,Y,Z are MSs.

1. If f,g are continuous on their respective domains, g0f is continuous on X
2. For any subset A() of X, the restriction function f|A:AY is continuous whenever f is continuous
3. f is continuous iff f : Xf(X)(f(X) considered as a subspace of Y) is continuous.

**Continuity and Compactness**

It is known that a subset A of Ru  is compact iff any one of the following conditions are satisfied: (1) A is bounded and every continuous real valued function on A is uniformly continuous, (2) every real valued continuous function on A is bounded. Although condition (2) characterises compact subsets of arbitrary MS, condition (1) do not: let X be an uncountable set with discrete metric d; Any real valued function on an infinite subset A of (X,d) is uniformly continuous and A is bounded but A is not compact.

**Theorem 6.8** Let X and Y be MSs, f:XY be continuous. If A (X) is compact, then f(A) is compact.

**Proof**  Let be an open cover of f(A) in Y: f(A). Then Af-1(=; thus is an open cover of A. Since A is compact, A has a finite subcover {}: A ; thus f(A). Thus f(A) is compact.

**Note** If a function maps compact sets to compact sets, it does not necessarily follow that f is continuous: consider f:RR, f(x)=1, if x is rational; =0, if x is irrational, maps every subset of R to a subset of {0,1}, which is obviously compact though f is not continuous.

**Theorem 6.9** Let (X,d) be compact MS, f :(X,d)Ru be continuous. Then f is bounded (that is, f(X) is a bounded subset of R) and there exist c,dX such that f(c)= min{f(x): xX} and f(d)=max{f(x): xX}.

**Proof** Since X is compact, f(X) is compact and hence closed and bounded subset of R. Thus sup f(X) and inf f(X) exist as real numbers and are limit points of f(X); hence sup f(X)f(X) and inf f(X)f(X). Thus there exist c,dX such that f(c)= inf{f(x): xX} and f(d)=sup{f(x): xX}. Since inf and sup of the set {f(x): xX}belong to the set, they are the respective minimum and maximum element of the set.

**Note** If f is a bijective continuous function from a MS to a MS, it is not necessary that f-1 need be continuous. For example, the identity map from (R,d) (d is the discrete metric) to Ru is continuous and bijective but the inverse map is not continuous. However if the domain is compact, then inverse function is continuous:

**Theorem 6.10** Let X and Y be MSs, X compact, f:XY be continuous and bijective. Then f-1: YX is continuous and f is a homeomorphism.

**Proof** Let C be a closed set in X. Then C is compact since X is compact. Thus (f-1)-1(C)=f(C) is compact subset of Y and hence is closed subset of Y. Thus inverse image of any closed set in X under f-1 is closed in Y. Hence f-1 is continuous.

**Uniform Continuity**

f: (X,dX)(Y,dY) is continuous at cX iff for any >0, there exists (depending ,in general, on c and ) such that dX(x,c)< implies dY(f(x),f(c))<. In particular, if f:RuRu , f is continuous at cR iff for any >0, there exists (depending ,in general, on c and ) such that < implies <. Let us explain through an example that , in general, depends not only on but also on the point c under consideration. Consider g: RuRu, g(x)=x2. Then with =2, the statement ⇒ holds for c=1 but does not hold for c=10.

[ Justification: If <, then <x< and <<2 for c=1.

At c=10, for x=10, < but g(x)-g(10)=( 102 -102=5 and so >2]]

Thus even though g is continuous at 10 as well as at 1, for =2, the number =1/2 is usable at c=1 but unusable at c=10.

It is not difficult to show that , in fact, there is no one >0 such that the statement ⇒ holds for all real c: if there were any such >0, then for c>0 and x=c+,

2>

which would imply a<2 for all a>0, which is false. Hence, for this function g, corresponding to =2, there is no >0 that will work for all c simultaneously though g is continuous at each real c.

Consider , on the other hand, f:[0,1]R, f(x)=x2 . Let >0 be given. Let c[0,1]. ⇐=, independent of c and dependent on ONLY [Note: x,c[0,1] and hence ].

Thus, the property of uniform continuity of a function depends on its domain. Clearly, if a function is uniform continuous on its domain, it is continuous on that domain; converse may not hold. But if the domain of a continuous function is compact, then the function is uniformly continuous on the domain.

**Definition 6.3** Let f:X Y, X,Y are MSs. f is uniformly continuous on X iff for every >0, there exists >0 ( depending only on such that dX(x1,x2)< implies dY(f(x1),f(x2))< holds.

**Theorem 6.11**  Let f:XY, X,Y are MSs, f continuous on X,X compact. Then f is u.c. on X.

**Proof** Let >0 be given. Let aX. f is continuous at a. Corresponding to chosen >0 and for aX, there exists such that dX(x,a)< implies dY(f(x),f(a))<. Consider the collection of open balls U={B(a,): aX}. Obviously U is an open cover of the compact MS X. Let {B(ai,): i=1,…,n} be a finite subcover of U. Let =min{: i=1,…,n}>0. Let x,yX with dX(x,y)<. Let xB(ak,) (k is fixed and from {1,…,n}). Then dX(x,ak) dX(x,y)+ dX(y,ak)<++=. yB(ak,). Also xB(ak,) B(ak,). Thus dY(f(x),f(ak))< and dY(f(y),f(ak))<. Hence dY(f(x),f(y)) dY(f(x),f(ak))+ dY(f(ak),f(y))< . Hence for given >0, >0 has been found such that dX(x,y)< implies dY(f(x),f(y)) < holds.

**Corollary** If f:[a,b] R is continuous on [a,b], then f is u.c. on [a,b].

**Theorem 6.12** Let f:XY, X,Y are MSs, f uniformly continuous on X. If (xn) be a CS in X, then (f(xn))is Cauchy in Y.

**Proof** Since f is u.c. on X, corresponding to >0, there exists(depending only on )>0 such that dX(x,y)< implies dY(f(x),f(y))< holds. Since (xn) is Cauchy, corresponding to >0, there exists natural m such that p,qm implies d(xp,xq)< . Hence dY(f(xp),f(xq))< for all p,qm. Hence (f(xn)) is Cauchy.

Continuous image of a bounded space may not be bounded: f :(0,1)R, f(x)=1/x. Towards this, we have

**Theorem 6.13** Let f:E(Ru)Ru be uniformly continuous on E and E is bounded subset of Ru. Then f(E) is bounded in Ru.

**Proof** We first prove that every sequence in f(E) has a convergent subsequence. Let (f(xn))be a sequence in f(E). Then (xn) is a sequence in the bounded subset E and hence is bounded. Thus (xn) has a convergent subsequence, say, ( . ( is Cauchy and since f is u.c. on E, (f( is Cauchy and hence convergent subsequence in Ru. Next, if possible, f(E) be not bounded, say, unbounded above (unbounded below case can similarly be handled). Take y1f(E), y2f(E) with y2>y1+1, y3f(E) with y3>y2+1, and so on. Then the sequence (yn) in f(E) has no convergent subsequence, contradiction. Hence f(E) is bounded.

**Note**

1. If f:E(Ru)Ru be uniformly continuous on E and E is unbounded subset of Ru, f(E) need not be bounded in Ru: the function f(x)=x is uniformly continuous on E but unbounded on E.
2. If E is bounded but noncompact subset of Ru, then E has a limit point a, not belonging to E, and then the function f:ERu ,f(x)= is continuous but unbounded on E and hence can not be u.c. on E by result above.

**Continuity and Connectedness**

**Theorem 6.14** Let (X,dX) and (Y,dY) be MSs, AX, f:XY be continuous on X. If A be connected in X, then f(A) is connected in Y.

**Proof** If f(A) is not connected in Y, then there exist nonempty disjoint subsets U,V of f(A), U,V are open in f(A), such that f(A)=UV. As U,V are open sets in f(A), there exist open sets U1,V1 in Y such that U=U1f(A) and V=V1f(A). By continuity of f on X, f-1(U1) and f-1(V1) are open in X, f-1(U)⊇f-1(U1)A, f-1(V) )⊇f-1(V1)A. Now f(A)= UV⇒A=]=]. Now A has been expressed as union of two sets which are open in A. Now , U1f(A) implies . Similarly . Also U1V1f(A)=UV= implies [][]=. Thus A b=is disconnected, a contradiction.

**Theorem 6.15** A MS (X,d) is disconnected iff there exists a continuous function from X onto the discrete space{0,1}.

**Proof** Let X be disconnected. Then there exist two open, disjoint, nonempty subsets A,B of X such that X=AB. Define f:X{0,1} by f(x)=0,for xA and f(x)=1, for xB. f is continuous on X.

Conversely, if f: X{0,1} is continuous on X, then f-1{0} and f-1{1} is a separation for X.

**The special case Ru**

**Theorem 6.16** Let f:[a,b]R be continuous on [a,b], f(a)f(b)<0 (that is, f(a) and f(b) are of opposite sign). Then there exist c(a,b) such that f(c)=0.

**Proof** Since [a,b] is connected and f is continuous on [a,b], f([a,b]) is connected subset of Ru. Hence f([a,b]) is an interval in R. Without loss of generality, let f(a)>0 and f(b)<0. Since f(a),f(b) f([a,b]) and f(b)<0<f(a) and since f([a,b]) is an interval, 0 f([a,b]). Thus there exists c [a,b] such that f(c)=0. Since none of f(a) and f(b) is zero, ca,b. Thus there exist c(a,b) such that f(c)=0.

**Corollary** (Bolzano’s Theorem) Let f:[a,b]R be continuous on [a,b], f(a)f(b) and let k be a real number between f(a) and f(b). Then there exists c(a,b) such that f(c)=k.

**Proof** W.O.L.G. let f(a)<k<f(b). Define g:[a,b]R by g(x)=k-f(x). g is continuous on [a,b], g(a)>0, g(b)<0. By previous result, there exist c(a,b) such that g(c)=0, that is, f(c)=k.

**Theorem 6.17** Let (X,d) be a connected MS with X countable. A real-valued function f on X is continuous iff f is constant.

**Proof** Let f : (X,d)Ru be continuous where X is countable and connected. Then f(X) is connected and countable. Since a connected subset of Ru is either or singleton or an interval, f(X), being countable, is singleton, that is, f is constant.

Converse part is obvious, since a constant function is continuous.

**Theorem 6.18** Every connected subset(with at least two points) of a MS is uncountable.

**Proof** Let A be a connected subset of a MS (X,d) such that A contains at least two distinct points a,b. Define f:A R by f(x)=d(x,a). Then f : (A,dA) Ru is continuous. Since A is connected , f(A) is connected subset of R. Since f(A) contains two distinct elements f(a)=0 and f(b)0(say, f(b)>0), f(A) is an interval in Ru. Thus f(A) must contain the interval [0,f(b)]. Since [0,f(b)] is uncountable, so is f(A). Hence A is uncountable.

**Homeomorphism and Isometry**

**Definition 6.4** A function f :(X,dX) (Y,dY) is a homeomorphism iff f is bijective and both f and f-1 are continuous.

**Definition 6.5** A function f :(X,dX) (Y,dY) is an isometry iff f is bijective and dX(a,b)=dY(f(a),f(b)), for all a,bX.

**Note** An homeomorphism f :(X,dX) (Y,dY) establishes a bijective correspondence between the sets X and y as well as between the collection of open sets of (X,dX) and (Y,dY). Hence properties of MS (like separability, compactness, connectedness)which can be framed in terms of open sets only are preserved under homeomorphism: such properties of MS are called **topological property** or topological invariant. But there are properties of MS like boundedness, completeness etc. which are not completely governed by the collection of open sets in the MS but which remain invariant under isometry: such properties are called **metric property** or metric invariant.

**Theorem 6.19** An isometry f: (X,d1)(Y,d2) is a homeomorphism; thus every topological property is a metric property.

**Proof** For any aX and any r>0, f[(a,r)]={f(x):d1(a,x)<r} ={f(x):d2(f(a),f(x))= d1(a,x)<r}=(f(a),r) (since f is bijection, yY iff y=f(x) for some xX). Let V be open in (Y,d2) and af-1(V). f(a)V. Hence there exists r>0 such that (f(a),r)V. As shown above, f[(a,r)]=(f(a),r)V; hence (a,r)f-1(V). Thus f-1(V) is open in (X,d1), proving continuity of f. Continuity of f-1 can similarly be established.

**Note** The converse is false: On R, define d1(x,y)=. It can be verified that Ru is topologically equivalent to (R,d1). Thus the identity map 1R:Ru(R,d1) is a homeomorphism but 1R  is not an isometry: d1(1R(1), 1R(2))=d1(1,2)=.

Since an isometry preserves distances, the property of being of finite diameter is a metric invariant, but is not a topological invariant: Ru is topologically equivalent to (R,d1), Ru is of infinite diameter while R in (R,d1) has diameter 1.

Example 6.3 Let S={} with usual metric induced from Ru and T=N with usualmetric. Let f:ST , f(n)=. F is a homeomorphism (every element of S and T are isolated points). S and T are not metrically equivalent since S is bounded and T is not so. Also T is complete while S is not so.

Example 6.4 Consider (-1,1) as a subspace of Ru and define f : (-1,1)Ru by f(x)=. Then f is bijective continuous function. Also the inverse g( of f) is given by g(x)= which is also continuous. Hence (-1,1) and Ru are homeomorphic.

**Some Problems**

1. Prove that f:XY is continuous iff re
2. Prove that every closed subset C in a MS (X,d) is a -set.

Let ={B(c,):cC}, n natural. To prove that C={:n natural}.

1. Show that f : (X,d1)Y,d2) is uniformly continuous iff whenever d1(xn,yn) tends to 0, we have d2(f(xn),f(yn)) tends to 0 . Use this to show that f (x)=x2 is not u.c. on R.
2. In a MS (X,d), let the sequence (xn) have a limit x. Then the set containing all the terms of the sequence and the limit of the sequence is a compact set.

Proof of (3) Let A={xn}{x}. Take any sequence (ak) from A: if ak=x for infinitely many k, then a subsequence of (ak) each term of which is x can be formed which therefore converges to xA. Otherwise (ak) is a subsequence of (xn). Since x=lim(xn), x=lim(ak). Combining the two cases, we see that we can extract a convergent(in A) subsequence from the terms of any sequence (ak)in A. Hence A is sequentially compact and hence compact.

**Continuous functions in Ru: some more results**

**Theorem 6.20** Let f : (a,b)Ru be continuous on (a,b). f is uniformly continuous on (a,b) iff and exist as real numbers.

**Proof** Let =L and =M exist. Define g:[a,b]Ru by g(x)=f(x), a<x<b; g(a)=L, g(b)=M. Then g is continuous on (a,b); also =L=g(a) and =M=g(b). Hence g is continuous on compact [a,b]. Thus g is u.c. on [a,b]. Hence g, that is, f is u.c. on (a,b).

Conversely, let f be u.c. on (a,b).Thus for any >0, there exists >0 such that x,y(a,b), implies Let (xn) be a sequence in (a,b) such that lim(xn)=a in Ru. Hence (xn) is a C.S. in Ru and hence is a CS in (a,b). Since f : (a,b)Ru is u.c., (f(xn)) is Cauchy and hence convergent in the complete MS Ru. Let lim (f(xn))=L. If (yn) be another sequence in (a,b) such that lim(yn)=a in Ru, then lim(xn-yn)=0 in Ru. Corresponding to >0, there exists natural number k such that nk implies . Thus for nk, holds. Hence lim(f(yn))=lim(f(xn))=L. Thus for any sequence (xn) in (a,b) with lim(xn)=a, lim(f(xn))=L holds. Hence, from sequential definition of limit, =L. Similarly we can show that exist as real number.

Example 6.5 f(x)=1/x is continuous on (0,1) but not u.c. on (0,1) since does not exist as real number.

**Note** The function g constructed from f in above result is called continuous extension of f to [a,b]. As seen in the above example, continuous extension of f :(a,b)Ru may not exist.

**Theorem 6.21**  Let f :(a,b)Ru. f admits of a continuous extension to R iff f is u.c. on (a,b).

**Proof** Let f be u.c. on (a,b). Thus and exist as real numbers. Define g:RuRu by g(x)=L, xa ; g(x)=f(x), a<x<b; g(x)=M, xb. g is continuous at a since ==L=g(a); g is continuous at b since ==M=g(b). Thus g:RuRu is continuous extension of f to R.

Conversely, let f possesses a continuous extension g:RuRu. Then g is continuous on compact [a,b] and hence is u.c. on [a,b]; since g and f agree on (a,b), f is u.c. on (a,b).

**Lipschitz Condition**

f:[a,b]R satisfies Lipschitz condition on (a,b) iff there exists M>0 such that , for all x,y in (a,b).

**Theorem 6.22**  If f obeys Lipschitz condition on (a,b), then f is u.c. on (a,b).

**Proof** Let >0 be given. Let =/M. x,y(a,b), ⇒. Hence the result.

Example 6.6Let f:RR satisfies f(x+y)=f(x)+f(y) for all x,y in R. Let f be continuous at cR. Show that f is u.c. on R.

**Continuity and Monotonicity**

**Theorem 6.23**  Let f:[a,b]R be continuous and injective. Then f is monotonic.

**Proof** Since ab, f(a)f(b).W.L.O.G. let f(a)<f(b). First we show a<x1<b imply f(a)<f(x1)<f(b). If the statement is false, then either (A) f(x1)<f(a)<f(b) or (B) f(a)<f(b)<f(x1) holds.

Case A. If f(x1)<f(a)<f(b), since f is continuous on [a,b], by Intermediate Value Theorem, there exists x/ (x1,b)(a,b) such that f(x/)=f(a), contradicting injectivity of f on [a,b].

Case B. If f(a)< f(b)<f(x1), similarly there exists x// in (a,x1) such that f(x//)=f(b), contradiction.

Thus a<x1<b imply f(a)<f(x1)<f(b). (\*)

Next, let a<x1<x2<b. To prove f(x1)<f(x2). Applying (\*) to f on[a,x2] , f(a)<f(x1)<f(x2). Again, Applying (\*) to f on[x1,b] , f(x1)<f(x2)<f(b). Combining , f(a)<f(x1)<f(x2)<f(b), proving that f is strictly increasing on [a,b]. Similarly, if f(a)>f(b), f can be proved to be strictly decreasing.

**Theorem 6.24** Let f:[a,b]R be continuous and strictly monotone (increasing on [a,b] or decreasing on [a,b]). Then f-1: Range(f)[a,b] is continuous and strictly monotone increasing.

**Proof** To concretize, let f be strictly m.i. and continuous on [a,b] : f([a,b])=[f(a),f(b)]. Since f is continuous and bijective on the compact set [a,b], f-1:[f(a),f(b)] [a,b] is continuous on [f(a),f(b)]. Next we prove that f-1 is strictly m.i. Let y1,y2[f(a),f(b)], y1<y2. Since f is bijective, there exist unique x1,x2 [a,b] such that y1=f(x1), y2=f(x2). Since f is strictly m.i. and f(x1)=y1<y2=f(x2), it follows that f-1(y1)=x1<x2=f-1(y2)( x1>x2 implies y1=f(x1)>f(x2)=y2, contradiction.

**Lemma 6.1** Let f be m.i. on (a,b) and let c(a,b). Then (1) f(c-0)=sup{f(x):x(a,c)}, (2) f(c+0)=inf{f(x):x(c,b)}, (3) f(c-0)f(c)f(c+0).

**Proof** If x(a,c), then a<x<c implies f(x)f(c). Thus sup{f(x):x(a,c)} f(c). Let b=sup{f(x):x(a,c)}. For an arbitrary >0, there exists x0 in (a,c) such that b-<f(x0)<b. If x0<x<c, then b-<f(x0)f(x)b<b+. Thus x0<x<c implies . Thus f(c-0)=b f(c). Similarly f(c) f(c+0). Combining, f(c-0) f(c)f(c+0).

**Note** Let f be m.i. on (a,b) and let a<c<d<b. Then f(c+0) f(d-0): Let x0 satisfy c<x0<d. Then f(c+0)= inf{f(x):x(c,x0)}f(x0)sup{f(x):x(x0,d)}=f(d-0).

**Note** Monotonic functions can only have (if at all)discontinuity of first kind (that is, f(c+0) and f(c-0) both exist but may not be equal at a point c in the domain of definition of f)but notdiscontinuity of second kind.

**Theorem 6.25** Let f be monotonic on (a,b). The set of points of discontinuity of f is at most countable.

**Proof** To concretize, let f be m.i. Let T=(x: x(a,b),J(x)=f(x+0) – f(x-0)0} be the set of all points of discontinuities of f. Let c,d (a,b), c<d. As proved above, f(c+0)f(d-0). The intervals (f(c-0),f(c+0)) and (f(d-0),f(d+0)) are disjoint . Let xT. Choose rational number rx(f(x-0),f(x+0)). Define :TQ by (x)=rx. If x1x2, say x1<x2, then as seen above (f(x1-0),f(x1+0)) and (f(x2-0),f(x2+0)) are disjoint intervals containing x1 and x2 respectively and hence (x1)==(x2). Thus is injective. Now (T) is a subset of the countable set Q of all rationals; hence (T) is at most countable. Now g:T(T) , g(x)=(x), x T, is bijective; hence T is at most countable.

**Banach’s Fixed Point Theorem**

**Theorem 6.26** Let (X,d) be a complete MS, f:(X,d)(X,d) be a contraction mapping[ that is, there exists fixed t, 0<t<1, such that f(f(x),f(y))t d(x,y) for all x,y in X]. Then f has a unique fixed point, that is, f(x)=x has a unique solution.

**Proof** Let >0 be given. Then d(f(x),f(y))<⇐t d(x,y)<⇐d(x,y)<=(say); thus f is uniformly continuous and hence continuous on (X,d). Let xX. We shall prove that the sequence (xn) defined by xn=fn(x),nN, is a CS in (X,d). Now , for m>n, d(xn,xm)=d(f(xn-1),f(xm-1))t d(xn-1,xm-1)t2 d(xn-2,xm-2)…tnd(x,xm-n)tn[d(x,x1)+d(x1,x2)+…+d(xm-n-1,xm-n)]tn[d(x,x1)+t d(x,x1)+…+tm-n-1d(x,x1)]=tnd(x,x1)[1+t+t2+…+tm-n-1]= tnd(x,x1)< which tends to 0 as n and hence m. Hence (xn) is a CS in complete MS (X,d); thus lim(xn)=x0X. We shall next prove that x0 is a fixed point. f(x0)=f(lim(xn))=lim(f(xn)) (since f is continuous)= lim(xn+1)=x0. To prove uniqueness of x0, suppose f(y0)=y0, x0 y0. Then d(x0,y0)=d(f(x0),f(y0)) t d(x0,y0). Since d(x0,y0)0, 1 t but 0<t<1 from definition, contradiction. Hence x0=y0.

**Note** Picard’s Existence Theorem of solutions for first order DE uses Banach’s fixed point theorem.

Example 6.7 Let T:l2l2, T((xn))=(xn/2) is a contraction map since d(T((xn)),T((yn)))==d((xn),(yn)).

**Lemma 6.2** Let (X,d) be a MS, A⊆X. Define f:(X,d)Ru by f(x)=d(x,A)=inf{d(x,a):aA}. Prove that f is continuous on X.

**Proof** Let x0X. To prove : f is continuous at x0, that is, for every >0, there exists >0 such that d(x,x0)< implies . Let =/2.

For aA and xB(x0,/2), d(x,a)d(x,x0)+d(x0,a)</2+ d(x0,a); thus inf{d(x,a):aA}/2+ inf{d(x0,a): aA}, implying d(x,A) /2+d(x0,A). Hence d(x,A)-d(x0,A)/2<; similarly d(x0,A)-d(x,A)<; combining, <

**Theorem 6.27** (Urysohn’s Lemma) Let (X,d) be a MS, A and B are disjoint closed subsets of X. Then there exists f:(X,d) Ru such that f is continuous, 0f(x)1 for all x in X, f(A)={0} and f(B)={1}.

**Proof** Define g :(X,d) Ru by g(x)=; note that =0 implies =0=, which implies x=AB=, contradiction.Since d(x,A),d(x,B)0, we have 0g(x)1 for all x in X. Obviously, g(A)={0}. For bB,b∉A=, hence d(b,A)>0, d(b,B)=0 and hence g(b)=1. By Lemma above, g is continuous on X.

**SOME MORE PROBLEMS**

1. (xn)x iff (x1,x,x2,x,x3,x,…)x

≻ Let (xn)x. Define (yn) by : yn=x, for n even; yn=xn-2, for n odd >3. Since (xn)x, corresponding to >0, there exists natural m such that d(xn,x)<, for all nm. For n2m+1,

n even implies d(yn,x)=d(x,x)=0<

n odd implies d(yn,x)=d(xn-2,x)<, since n2m+1 implies n-22m-1>m.

Thus (x1,x,x2,x,x3,x,…)x.

Conversely, if (x1,x,x2,x,x3,x,…)x, its subsequence (xn)x.

1. Let f,g:(X,d1)(Y,d2) be continuous functions on X; let A={x:f(x)=g(x)}. Prove that A is closed in (X,d1).

Let aX-A={x:f(x)g(x)}. Then f(a)g(a); hence there exists r>0 such that B(f(a),r)B(g(a),r)=.Then af-1[B(f(a),r)]g-1[B(g(a),r)] (an open set in (X,d) since f,g are continuous)X-A {xf-1g-1f(x) and g(x); hence f(x) g(x)}. Hence X-A is open and hence A is closed.

1. Let f :(X,d)(X,d) be continuous . Let A={xX: f(x)=x} (that is, x is a fixed point of f). Then A is closed which follows from result just above taking g as 1A.
2. Let (X,d) has no isolated point and AX. Let each a in A is an isolated point of (A,dA). Prove that A is NWD in (X,d).

1.Justify whether true or false: If f is an injective continuous function between two MSs X and Y and if B is an open ball in X, then f(B) is an open ball in Y.

Let f:RdRu be defined by f(x)=x, where Rd is set of all real numbers with discrete metric. f is continuous and injective but f(B(x,1/2))=f({x})={x} is not an open ball in Ru.

2. If f:RuRu be continuous and if range of f is a proper superset of N, then f-1(N) is not open unless empty.

Since N is closed in R and f is continuous, f-1(N) is closed. If f-1(N)is open and if f-1(N) then Ru is disconnected, contradiction. Note that f-1(N)R by given condition.

3. Let G be an open subset of R. Prove that the characteristic function of G, fG :RR defined by fG(x)=1, if xG and =0, if xR-G, is continuous at each point of G.

Let gG. Let (xn) be a sequence in R converging to g. Since G is open, there exists positive integer m such that xnG for nm.Thus fG(xn)=1 for nm. Hence (fG(xn))1=fG(g). Hence fG is continuous on G.

4. Give an example of subsets A and B of R2 such that all three of the following conditions hold:

(1) neither A nor B is open, (2)AB=,(3)AB is open

5. Show that Ru and Rd are not homeomorphic.

Let f:Rd be a homeomorphism. Let yR. {y} is open in Rd but f-1{y}, a singleton in Ru, cannot be open in Ru, contradiction.

6. Give an example of a MS X such that every subset of X is totally bounded.

A finite discrete MS X={a1,…,an}. For 0<<1, {a1,…,an} is an -net. For 1, {a1} is an -net.

7. Prove that a connected subset of Rd is compact.

Every connected subset of Rd is singleton and hence is compact.

8. A is compact subset in a MS (X,d) and b is a limit point of A. Prove that there exists aA such that d(a,b)=d(A,b) .

A compact implies A is closed which implies bA. Thus d(A,b)=0=d(b,b). We may take a as b.

9. A,B are disjoint compact sets in a MS X. Prove that d(A,B)>0. Show that there exist disjoint open sets O1 and O2 such that AO1 and BO2.

Fix aA. For every bB, there exist open sets Oa, Ob such that aOa, bOb, OaOb=. {Oa:aA} is an open cover of compact set A; hence there exists finite subcover {,…,}: A⊆=OA (say). Let Ob=. Clearly OAOb=. Allowing b to vary over B and using compactness of B, there exists open nonempty OB such that OAOB=, A⊆OA, B⊆OB.

If d(A,B)=0, then for = ( n natural), there exists anA, bnB such that d(an,bn)<. Since A,B are sequentially compact, (an) has a subsequence (aA, (bn) has a subsequence (bB. Thus d(,)<. Thus for large enough k, d(a,b)d(a,)+d(,)+d(,b)<0 as k. Thus d(a,b)=0, contradicting A, B are disjoint.

10. Prove that from any infinite open covering of a separable MS, we can extract a countable subcovering.

11. If f,g:XR are continuous, then the set A={xX:f(x)>g(x)} is open in X.

A=(f-g)-1(0,) is open in X.

12. Let X,Y be MSs, f:XY, A,B are open sets in X. Prove that if f is continuous on A and on B, then f is continuous on AB. Is this result true if A and B are closed?

Let U be open in Y. Since f is continuous on A and on B, f-1(U)A is open in A and since A is open in X, f-1(U)A is open is open in X and hence in AB; similarly f-1(U)B is open in AB. Thus f-1(U)AB]=[ f-1(U)A][ f-1(U)B] is open in AB. Hence the result.

13. f:RR be the continuous mapping f(x)=. Find (1) an open set O such that f(O) is not open, (2) a closed set F such that f(F) is not closed, (3) a set A such that f(.

(1) f(-1,1)=[0,1/e), (2) f(R)=(0,1/e], (3)for A=(0,), =[0,), f(=(0,1].

14. Let (X,d) be a MS , A,B are subsets of X such that d(A,B)>0. Prove that A,B are separated; converse may not hold.

Let 0<k=d(A,B)d(a,b), for all aA, bB. Thus B(a,. Hence a does not belong to Thus A. Similarly B. Converse may not hold: let A=(0,), B=(-,0).

15. Two open sets in a MS are separated iff they are disjoint.

If A,B are open, A, then A⊆X-B, closed; hence ⊆X-B and hence A. Similarly B. Thus A and B are separated.

Conversely, if A,B are separated, then B.

16. If an open (closed) set O=AB, where A,B are separated, then A,B are open(closed resp)

Let aA. Then a∉. Thus there exists r>0 such that B(a,r)B=. B(a,r)BO=. Hence B(a,r)O⊆A. Thus A is open in O and hence in MS X.

If O is closed, O=AB, AB, then O==,hence ⊆A; thus A is closed. Similarly B.

17. A,B are non-empty sets in a MS X and B is compact. Prove that d(A,B)=0 iff B.

18. Prove that (0,) is not homeomorphic to (0,1).

19. Give an example of an open cover of (0,1) which has no finite subcover.

.

20. Let Q be the MS of rational numbers under the metric d(p,q)=. Let E={pQ:2<p2<3}. Show that E is closed and bounded in Q but not compact. Is E open in Q?

E=[: so E is closed in Q. EB(0,2): hence E is bounded in Q. is an open cover of E having no finite subcover. E=( is also open in Q.

21. Construct a compact set of real numbers whose limit points form a countable set.

Let E1={1}, E2={1+,…,Em={1+. Consider E=.

22. Give examples to show that for subsets A,B of a MS(X,d), need not imply A=B, A0=B0 need not imply A=B, int (cl A) need not be equal to cl(int A).

In Ru, consider subsets Q, R-Q.

23. Let A be a subset of a MS X such that d(A)<r and A. Prove that A⊆B(x,2r).

Let x1­­AB(x,r). Let y­­A. Then d(x,y)x1)+d(x1,y)<r+ d(x1,y)r+d(A)<2r. Hence yB(x,2r).

24. Show that for a subset A of a MS X, following are equivalent: (1) A is dense in X, (2) F is closed in X and AF implies F=X, (3)int(X-A)=.

(1)⇒(3) Since A is dense in X, every nonempty open set in X intersect A; thus X-A cannot contain any nonempty subset; thus int(X-A)=.

(3) ⇒(2) Let F be closed in X and AF. Thus X-F is open and contained in X-A. Hence int(X-A) unless X-F=.

(2) ⇒(1) Let U be open in X and U. Then A⊆X-U, X-U closed. So X-U=X. Thus U=.Hence A is dense in X.

25. Let aQ and bR-Q, where a,b>0. Show that {xQ:-a<x<a} is open but not closed in the MS Q (with subspace topology)while {xQ:-b<x<b}is clopen in Q.

{xQ:-a<x<a}=(-a,a)Q is open in Q. a is a limit point of {xQ:-a<x<a}in Q which does not belong to {xQ:-a<x<a}.Hence it is not closed in Q. {xQ:-b<x<b}=[-b,b] Q=(-b,b)Q: hence clopen in Q.

26.If A and B are two dense subsets of a MS (X,d), show that AB is dense but AB need not be dense.If one of A,B is, in addition, open , show that AB is dense.

In Ru, Q and R-Q are dense but =Q(R-Q) is not dense in Ru. If A,B are dense in (X,d), then =X; so AB is dense in X. Next, let A,B are dense in X and let A be open: to prove AB is dense in X. Let V be any nonempty open set in X; since A is dense in X, A. Since B is dense in X and A is a nonempty open set in X, B. Thus V; hence AB is dense in X.

27. Let (X,d) be a MS and Y be a separable and dense subset of X. Show that X is separable.

, so there is a countable subset A of Y such that clYA=Y. It suffices to prove that clXA=X. Let U be any nonempty open set in X. Since , UY. UY is a nonempty open set in (Y,dY). Since clYA=Y, we have U, that is, U. Thus clXA=X and A is separable.

28. Use Baire Category Theorem to prove that (1) the set of irrational numbers is not a counable union of closed subsets of R; hence show that there is no function f:RR having the irrational numbers as the set of its discontinuities. (2)the closed interval [0,1] is an uncountable set.

(1) Let Q={x1,x2,…} and if possible, let R-Q=, where each An is closed in Ru. Then R=.Since R is of second category and since int{xn}= for all n, so we must have int(An) for at least one n. Thus An contains an open interval I(say) and since An⊆R-Q, I cannot contain any rational number, absurd. The second part follows from the fact that the set of discontinuities of a function f:RR is an -set.

(2) Let [0,1] be countable: [0,1]={x1,..,xn,..}.Let An={xn}, n natural. Then int(cl An)= for all n and [0,1]=. Thus [0,1] is of first category, contradiction since [0,1] is a closed subset of the complete MS Ru and hence complete MS; thus is of second category.

29.Let f be a continuous injection from a MS (X,d) to a MS Y endowed with discrete metric d1. If f is continuous at aX, show that {a} is an open ball in X.

{f(a)}=B(f(a),1/2) is an open set in Y. Since f is continuous at a, there exists nbd. U of a in X such that f(U)⊆{f(a)}. Since f is injective, U={a}.

30. Show that for a real-valued function f on a MS (X,d) is continuous iff for all real a, {xX:f(x)<a} and {xX:f(x)>a} are open sets in X.

If f is continuous on X, {xX:f(x)<a}= f-1(- and {xX:f(x)>a}=f-1(a,) are open in X, for all real a. Conversely, for bX and r>0, f-1(f(b)-r,f(b)+r)=f-1(- f-1(f(b)-r,) is open and hence f is continuous at b; since b is arbitrary, f is continuous on X.

31.(Pasting Lemma) Let A,B are closed sets in (X,d) such that X=AB, let f:AY and g:BY be continuous maps such that f(x)=g(x), for all x in AB, where Y is a MS. Show that the pasting

map h:XY defined by h(x)=f(x), for xA; h(x)=g(x), for xB, is continuous.

Let C be closed in Y. Then h-1(C)=h-1(C)X= h-1(C)[AB]=[ f-1(C)[g-1(C)B] is closed in X. Since f(x)=g(x), for all x in AB, h is well-defined.

32. Prove that boundedness, total boundedness and completeness are not topological properties.

f :(0,1][1,, f(x)=1/x is a homeomorphism.(0,1] is totally bounded but f((0,1])=[1, is not bounded and hence not totally bounded. (0,1] is not complete whereas [1, is complete.

33. Let A is closed and B is compact in (X,d). Show that AB is compact.

AB is closed in compact B; hence.

34. If every proper closed subset of a MS X is finite, show that X is compact.

Let be an open cover of X; let . Then X- is closed and hence finite, say, X-={a1,…,an}. If ai(i=1,…,n), then {,} is a finite subcover of .

35. Show that a surjection f:[a,b]B, where B is a non-closed subset of R, cannot be continuous.

Let f:[a,b] B be a continuous surjection and B is not closed in R. Thus there exists y.Let (bn) be a sequence in B such that lim(bn)=y. Since f is surjective, there exists an in [a,b] such that f(an)=bn. Since [a,b] is compact and hence sequentially compact, (an) has a subsequence converging to m (say) in [a,b]. Then y=lim(bn)=lim(lim(f(=f(lim(=f(m)B, contradiction. Hence f is not continuous.

. On the set R of reals, consider the metric d given by d(x,y)=min{1,. Show that (R,d) is bounded and complete but is not totally bounded.

(R,d) be totally bounded and let A={x1,…,xn} be a ½-net for R. Let y>max{,…,}+1. Then , for all i. Thus d(xi,y)=1, for all i. Hence A is not ½-net. Thus (R,d) is not TB. (R,d) is obviously bounded. Let (xn) be a CS in (R,d). Then for every >0, there exists natural p such that d(xm,xn)< for all m,np. Thus < for all m,np. Hence (xn) is Cauchy in Ru which is complete. Thus there exists real x such that lim(xn)=x in Ru. Thus d(xn,x)< for all n, q natural. Hence (xn) is convergent in (R,d).

Let X and Y be MSs. Show that f:XY is continuous iff the restriction of f to every compact subset of X is continuous.

XY be continuous, then every restriction of f is continuous. Conversely, let f|C:CY be continuous for every compact subset C of X.Let xX and let (xn) be a sequence in X converging to x. Then A={x,x1,x2,…} is compact. Since f|A :AY is continuous, {f(xn)} f(x) in Y. Thus f is continuous.

38. Show that two sets A,B in a MS X are separated iff there exist open sets U,V such that A⊆U, B⊆V, A

Let A,B be separated. Then .Thus A⊆X-=U (say), B⊆X-=V(say). Also AV=A= Similarly, B

Conversely, let there exist open sets U,V such that A⊆U, B⊆V, A. Thus A⊆X-V, closed. Hence Thus and so . Similarly .

**CHAPTER VII**

**DIFFERENTIABILITY OF f: (a,b)R**

**Definition7.1** Let f: (a,b)R, c(a,b). f is differentiable at c iff exists and in that case, we write f/(c)=.

**Definition7.2** Let f: (a,b)R, c(a,b). f is increasing at c iff there exists >0 such that f(x)<f(c) for all x(c-,c) and f(x)>f(c) for all x(c,c+). f is decreasing at c iff there exists >0 such that f(x)>f(c) for all x(c-,c) and f(x)<f(c) for all x(c,c+).

**Theorem 7.1**  Let (a,b)R, c(a,b). f is differentiable at c and f/(c)>0. Then f is increasing at c.

**Proof** Given f/(c)=>0. Corresponding to =f/(c)/2>0, there exist >0 such that for -<h<, f/(c)- f/(c)/2<< f/(c)+ f/(c)/2, that is ,>0. For -<h<0, f(c+h)-f(c)<0 and for 0<h<, f(c+h)-f(c)>0. Thus f is increasing at c.

**Note** Similarly, if f/(c)<0, then f is decreasing at c.

**Theorem 7.2** (Darboux’s Theorem)Let f be differentiable in [a,b] and let f/(a) f/(b). For any k strictly between f/(a) and f/(b), there exists c(a,b) such that f/(c)=k.

**Proof** To concretize, let f/(a)<k<f/(b). Let g:[a,b]R ,g(x)=f(x)-kx. g/(x)=f/(x)-k, x[a,b]. g/(a)<0 and g/(b)>0. Since g is differentiable on [a,b], g is continuous on [a,b]. Thus g attains maximum and minimum values on [a,b]: let g(c1)=m1= min{g(x): x [a,b]}. Since g/(a)<0, there exists >0 such that g(x)<g(a) for a<x<a+. Thus g(a) min{g(x): x [a,b]}. Thus c1a. Similarly g/(b)>0 implies there exists >0 such that g(x)<g(b) for b-<x<b. Thus g(b) min{g(x): x [a,b]}. Thus c1b. Hence a<c1<b. If g/(c1)>0, then there exists >0 such that for c- <x<c, g(x)<g(c1)=m1= min{g(x): x [a,b]}, contradiction. Hence g/(c1) ≯0. Similarly g/(c1)≮0. Thus g/(c1)=0. Hence f/(c1)=k, a<c1<b.

**Theorem 7.3** If f:[a,b]R be differentiable on [a,b], then the derived function f/ cannot have jump discontinuity at any point of [a,b].

**Proof** Let for each c(a,b], exists and for each c[a,b), exists. It is sufficient to prove =f/(c) for each ca,b] and =f/(c), for each c[a,b).

Let ca,b] and let =L. We prove: f/(c)=L. If L< f/(c), let us choose >0 such that L+< f/(c). Since =L, there exists >0 such that L-<f/(x)<L+ for all x in (c-,c)(a,b]. Let d(c-,c)(a,b]. Then L-<f/(d)<L+<f/(c). By Darboux’s Theorem, there exists (d,c) such that f/()=L+. But (d,c) implies (c-,c)(a,b] and hence f/()<L+, contradiction. Hence L is not less than f/(c).

Next, let L>f/(c). Let us choose >0 such that L-> f/(c). Since =L, there exists >0 such that L-<f/(x)<L+ for all x (c-,c)(a,b]. Let d(c-,c)(a,b]. Then f/(c)<L-<f/(d). By Darboux’s Theorem, there exists (d,c) such that f/()=L-. But (d,c) implies (c-,c)(a,b] and hence f/()>L-,contradiction. Thus L is not greater than f/(c). Hence =L= f/(c). Similarly = f/(c). This completes the proof.

**Note** If a function f is differentiable on an interval I and if f/ is monotonic on I, then f/ is continuous on I (Since a monotonic function defined on an interval can only have discontinuity of first kind).

**Theorem 7.4** (Rolle’s Theorem)If f:[a,b]R is continuous on [a,b], differentiable on (a,b) and f(a)=f(b), then there exists c(a,b) such that f/(c)=0.

**Proof** Since f is continuous on compact [a,b], there exists c,d[a,b] such that f(c)=max{f(x):x[a,b]}=M and f(d)=min{ f(x):x[a,b]}=m (say) .

**Case 1** If M=m, then f(x)=m, for all x[a,b]; thus f/(x)=0, for all x(a,b). Hence the theorem hold .

**Case 2** Let M>m. Let f(a)=f(b)=k. Clearly M=m=k is not possible. Let Mk. Then f(c)f(a). Thus ca. Similarly cb. Thus a<c<b. By assumption, f/(c) exists. If f/(c)>0, then there exists >0 such that f(x)>f(c) for x(c,c+), contradicting f(c)=max{f(x):x[a,b]}. If f/(c)<0, then there exists >0 such that f(x)>f(c) for x(c-,c), contradicting f(c)=max{f(x):x[a,b]}. Hence f/(c)=0, a<c<b.

**Geometrical Interpretation of Rolle’s Theorem**

**Theorem 7.5** (Lagrange’s Mean Value Theorem) If f:[a,b]R be continuous on [a,b] and differentiable on (a,b), then there exists c(a,b) such that f(b)-f(a)=(b-a)f/(c).

**Proof** Consider the function g:[a,b]R, g(x)=f(x)+dx where c is a constant chosen in such a way that g(a)=g(b), that is, f(a)+da=f(b)+db, that is, d= (\*). If we choose value of c as given by (\*), then g satisfies all the conditions of Rolle’s Theorem on [a,b]; in that case, there exists c(a,b) such that g/(c)=0, that is, f/(c)=.

**Theorem 7.6** (Aternative form of Lagrange’s Mean Value Theorem)If f:[a,a+h]R be continuous on [a,a+h] and differentiable on (a,a+h), then there exists c(a,a+h) such that f(a+h)-f(a)=hf/(a+ch), 0<c<1.

Example 7.1 Applying Lagrange’s MVT on f(x)=sin x on [0,], sin=sin0+ cos(c), 0<c<1. Hence <.

**Geometrical Interpretation of Lagrange’s MVT**

**Note** Rolle’s Theorem can be used to prove existence of solutions of equations and inequations:INCOMPLETE: EXAMPLE?

**Theorem 7.7** Let f:[a,b]R be continuous on [a,b] and differentiable on (a,b) and f/(x)>0 for all x(a,b). Prove that f is strictly increasing on [a,b].

**Proof** Let ax1<x2b. Since f satisfies all the conditions of Lagrange’s MVT on [x1,x2], then there exists c(x1,x2) such that f(x2)-f(x1)=(x2-x1)f/(c)>0, that is, f(x2)>f(x1).

Similarly we can prove

**Theorem 7.8** Let f:[a,b]R be continuous on [a,b] and differentiable on (a,b) and f/(x)=0 for all x(a,b). Prove that f is constant on [a,b].

**Theorem 7.9** (Cauchy’s Mean Value Theorem) If f,g:[a,b]R be continuous on [a,b] and differentiable on (a,b) and let g/(x)0 for all x in (a,b). Then there exists c(a,b) such that .

**Proof** Let h(x)=f(x)+cg(x), x[a,b], where c is chosen such that h(b)=h(a), that is, c=-. We note that g(b)g(a); otherwise, g satisfies all conditions of Rolle’s Theorem so that g/(x)0 for some x in (a,b), contradiction. Now h satisfies all the conditions of Rolle’s Theorem on [a,b]. By Rolle’s Theorem, there exists c(a,b) such that h/(c)=0, that is, .

Example 7.2 f(x)=, g(x)=, [1,4]. Then , 1<c<4. Thus (after calculation) c=2.

Example 7.3 f(x)=ex, g(x)=x2, [1,2]. By Cauchy’s MVT, the equation 3ex-2e(e-1)x=0 has a solution in (1,2). We can also prove it by taking continuity of F(x)= 3ex-2e(e-1)x into account since F(1) and F(2) are of opposite signs.

Example 7.4 Let f be a function which is differentiable in a neighbourhood of c and let =L . Prove that f/(c)=L.

**Proof** There exists h>0 such that f is continuous on [c,c+h] and differentiable in (c,c+h). By Lagrange’s MVT, there exists d, 0<d<1, such that =f/(c+dh). Thus f/(c+0)===L. Similarly f/(c-0)=L and hence f/(c)=L.

Example 7.5 Let g=f/ in (a,b). Then g cannot have discontinuity of first kind in (a,b).(Derived functions cannot have discontinuity of first kind)

**Proof** Let a<c<b. Then g(c)=f/(c)==, if exists.

**Theorem 7.10** (Generalized MVT of Taylor) Let f:[a,a+h]R be such that (1)f(n-1) is continuous on [a,a+h], (2) f(n) exists in (a,a+h). Let p be a positive integer. Then there exists c,0<c<1, such that f(a+h)=f(a)+hf(1)(a)+f(2)(a)+…+f(n-1)(a)+ Rn, where Rn=f(n)(a+ch) is called Remainder after n terms.

**Note** If we take remainder after 1 term, we get Lagrange’s MVT for p=1.

**Proof** Consider the function g:[a,a+h]R given by g(x)=f(x)+(a+h-x)f(1)(x)++…++m(a+h-x)p, where m is chosen such that g(a)=g(a+h)=f(a+h). Thus f(a+h)=f(a)+h f(1)(a)+f(2)(a)+…+mhp….. **(1)**

g obeys all condition of Rolle’s Theorem on [a,a+h]. By Rolle’s Theorem, g(1)(a+ch)=0 for some c, 0<c<1.

Now g(1)(x)=f(1)(x)-f(1)(x)+(a+h-x)f(2)(x)-(a+h-x)f(2)(x)+…= f(n)(x)- mp(a+h-x)p-1. Thus g(1)(a+ch)=0 gives m=f(n)(a+ch), 0<c<1. Thus Rn=mhp=f(n)(a+ch).

**Special Cases**

When p=n, Rn=f(n)(a+ch) (0<c<1): Lagrange’s form.

When p=1, Rn= : Cauchy form.

If we take a=0, the corresponding Taylor’s finite series is called Maclaurin’s finite series:

f(h)=f(0)+hf(1)(0)+f(2)(0)+…+f(n-1)(0)+Rn, where Rn= f(n)(ch), 0<c<1.

**Use of Taylor’s Theorem for error estimation and for existence of solution**

Example 7.6 Consider f(x)=ex on [0,1]. f(1)=f(0)+f(1)(0)+f(2)(0)+ f(3)(c) (0<c<1). Thus e=1+1++ec (INCOMPLETE)

Example 7.7 Prove that ln(1+x)>x-, for all x>0.

**Method 1** Let F(x)=ln(1+x)-x+ , x>0.

F/(x)==>0 for x>0. Since F/ exists in [0,x], Lagrange’s MVT can be applied.

F(x)=F(0)+xF/(cx), 0<c<1. Thus F(x)=0+x.>0. Hence ln(1+x)>x-, for all x>0.

**Method 2** Let F(x) = ln(1+x). Then F(1)(x)=, F(2)(x)=, F(3)(x)=.

Thus F(3) exists in [0,x], x>0. Applying Taylor’s Theorem with remainder after 3 terms,

F(x)=F(0)+xF(1)(0)+F(2)(0)+F(3)(cx), 0<c<1.

Thus ln(1+x)=0+x-+> x-.

Example 7.8 Prove that 0<.

Let F(x)=ln(1+x). By Lagrange’s MVT, F(x)=F(0)+xF/(cx), 0<c<1.

ln(1+x)=0+, or, .Hence 0<c=-<1.

Example 7.9 Let f be a function with f(2) existing in [a,b] and let f(a)=f(b). Also there exists c(a,b) such that f/(c)>0. Prove that there exists d(a,b) such that f(2)(d)<0.

INCOMPLETE.

Example 7.10 Prove that 0<<1.

**Proof** Let f(x)=ex. f(x)=f(0)+xf/(cx), 0<c<1. Thus ex=1+xecx. ecx= . Hence cx=ln(ecx)=ln(. Thus 0<c=<1.

Example 7.11 Let f(x)=0 be a polynomial equation. Let c,d be two real roots , c<d. Prove that the equation f/(x)+g f(x)=0 must have a real root m, c<m<d, where g is fixed real n)=umber.

**Proof** Let F(x)=egxf(x). F(c)=0=F(d); F is continuous and differentiable. By Rolle’s Theorem, F/(x)=0, c<x<d. Thus gegxf(x)+egxf(1)(x)=0 or, gf(x)+f(1)(x)=0 must have a real root in (c,d).

**Taylor’s Infinite Series Expansion**

Consider Taylor’s finite series with remainder after n terms:

f(x)=f(0)+xf(1)(0)+f(2)(0)+…+f(n-1)(0)+Rn(x).

Let x=x1.

f(x1)= f(0)+x1f(0)+ f(2)(0)+…+ f(n-1)(0)+Rn(x1)= Sn(x1)+Rn(x1).

Consider the infinite series of real numbers f(0)+x1f(0)+ f(2)(0)+…+ f(n-1)(0)+…

The above series will converge iff its partial sum sequence (Sn(x1)) converges.

Let the sequence (Rn(x1)) tends to 0 as n tends to . Then lim(Sn(x1))=f(x1) and in that case we can write f(x1)= f(0)+x1f(0)+ f(2)(0)+…+ f(n-1)(0)++….

Above infinite series is called Taylor’s Infinite Series corresponding to f; this expansion is valid at those points x1 where derivative of every order of f exist and lim(Rn(x1))=0. For example, ln(1+2)=2- is not valid since the series on the right does not converge (.

**Infinite series expansion of ex**

Let f(x)= ex, x real. f has derivative of every order at every real x.

We can check the finite series expansion f(x)=1+x++…++,0<c<1.

Hence 0<ecx<ex. Thus, for fixed x, lim(Rn)=lim(=0 as n tends to since lim()=0.

Hence ex can be expanded in Taylr’s infinite series as

ex=1+x++…++…

**Infinite series expansion of sin**

In this case f(x)=sin x has derivative of every order at every real x. Also Rn=sin, 0<c<1. Since sin is a bounded quantity and since lim()=0, lim(Rn)=0. Hence for all real x, sin can be expanded in Taylor’s infinite series as sin x=x-

**Infinite series expansion of ln**

Let f(x)= ln(1+x). f(n)(x)=, x>-1.

***Case1*** 0.Rn(x)=. Since 0 and 0<c<1, 0cxx . Hence 1+cx>1. Thus <1. Hence 0= And lim=0. Thus lim=0.

***Case2*** -1<x<0. Rn (Cauchy)==(-1)n xn

Since 0<c<1, <. -1<x<0 implies –c<cx<0 implying 0<1-c<1+cx<1. Thus 0<<1. Hence =bounded quantity x . Thus lim=0.

Hence ln(1+x) can be expanded in Taylor’s infinite series in (-1,1].

**Infinite series expansion of (1+x)m, m fixed real.**

f(x)=(1+x)m. f(n)(x)=m(m-1)…(m-n+1)(1+x)m-n.

***Case1*** 0<1. Rn(x)== m(m-1)…(m-n+1)(1+cx)m-n=(1+cx)m-n

For n>m, =<1, <1 (since 1+cx>1) and lim(xn)=0. Hence lim(Rn(x))=0.

***Case2*** -1<x<0. Rn(x)= INCOMPLETE

Example 7.12 Prove that ex>1+x+.

From Taylor’s finite series with remainder after 4 terms, ex=1+x++,0<c<1. Since >0, result follows.

**Local Extrema and Points of Inflexion**

**Definition7.3** If DR and c is an interior point in D , then f:DR has a local minimum at c if there is >0 such that (c-,c+D and f(x)f(c) for all x in (c-,c+. f has a local maximum at c if there is >0 such that (c-,c+D and f(x)f(c) for all x in (c-,c+.

As we approach a local minimum on a curve from left , the graph decreases and the tangents have negative slopes whereas as we approach from right, the graph is increasing and the tangents have positive slopes; reverse is the case for approach to a local maximum. In case the tangent exists at a point of local extremum, it is necessarily horizontal, that is, it has slope zero. We have seen that if f:DR is differentiable at an interior point c of D , then the vanishing of f(1)(c) is a necessary condition for f to have a local extremum at c. f:RR, f(x)=x3 at x=0 show that the condition is not sufficient to guarantee a local extremum. Above observations about the behavior of the graph lead to the following set of sufficient conditions for existence of extremum:

**Theorem 7.10(First order derivative test for local minimum)** Let DR , c is an interior point in D and f:DR be continuous at c. Then if f is differentiable on (c-r,c) (c,c+r) for some r>0 and if there is >0 with r such that f(1)(x)0 for all x in (c-,c) and f(1)(x)0 for all x in (c,c+), then f has a local minimum at c.

**Theorem 7.10(Second order derivative test for local minimum)** Let DR , c is an interior point in D. If f is twice differentiable at c and satisfies f(1)(c)=0 and f(2)(c)>0, then f has a local minimum at c.

Example 7.13 Examine the function f(x)=for local extremum.

f(x)=3-2x for x<1;=1, for 1<x<2;=2x-3, x>2. F is not differentiable at 1,2. For ½<x<1, f/(x)<0 and for 1<x<3/2, f/(x)>0; thus minimum at 1.

The corresponding results for local maximum is similar.

**Note** 1. Apart from differentiability in an open interval about c , except possibly at c, the continuity at c is essential for applying first derivative test for local extremum. Thus the theorem can be applied to yield local minimum at 0 of f(x)=; but the theorem cannot be applied to g(x)=[x] at 0.

1. While the First Derivative Test and Second Derivative Test provide sufficient conditions for a local extremum, neither of them is necessary. Consider f: (-1,1)R , f(x)= x2,if 0<<1 and f(0)=-1. f has strict local minimum at 0 though f is not continuous at 0. Consider f:RR , f(x)=x4. Then f has local minimum at 0 but f(2)(0) is not positive.

**Definition7.4** Let DR , c is an interior point in D and f:DR. c is a point of inflexion for f iff there exists >0 with (c-,c+)D such that f is convex on (c-,c) and concave on (c,c+) or vice versa.

Example 7.14 0 is a point of inflexion for f(x)=x3, xR.

**Theorem 7.11** Let DR , c is an interior point in D and f:DR. Then

1. (Necessary Condition for a Point of Inflexion) If f is twice differentiable at c and if c is a point of inflexion for f, then f(2)(c)=0.
2. (Sufficient conditions for a Point of Inflexion) Let f be thrice differentiable at c. If f(2)(c)=0 and f(3)(c)0, then c is a point of inflexion for f.

**Note** Condition in part (1) is not sufficient: Consider f:RR, f(x)=x4. 0 is not a point of inflexion though f(2)(0)=0. Condition in part (2) is not necessary: f(x)=x5, x real. f(3)(0)=0 though 0 is a point of inflexion for f.

More generally, we have

**Theorem 7.12**  let f be defined in (a,b) and a<c<b. Let f(1)(c)=…=f(n-1)(c) =0 but f(n)(c)0. Then n odd implies f has no extremum at c; if n be even, then f has an extremem at c and f(n)(c)<0 implies f has maximum at c and f(n)(c)>0 implies f has minimum at c.

**Proof** Let n be even.f(n)(c)has a real value 0. Thus f(n-1) is defined in a nbd. (c- of c. Let f(n)(c)>0. Then there exists >0 such that c-<x<c implies f(n-1)(x)<f(n-1)(c)=0 and c<x<c+ implies f(n-1)(x)>f(n-1)(c)=0. Let δ= min{Thus for δ >0 ,f(n-1) exists in (c- δ,c+ δ) and f(n-1)(x)<0 for c- δ<x<c and f(n-1)(x)>0 for c<x<c+ δ. Using Taylor’s finite series expansion

f(x)=f(c)+(x-c)f(1)(c)+…+=.

Thus f(c+ δ)-f(c)=>0 (since d(c,c+ δ)); that is , f(c+ δ)>f(c).

f(c)-f(c- δ)=-=)<0 where c- δ<g<c. Hence f(c+ δ)>f(c),f(c)<f(c- δ) for all sufficiently small δ. Thus f has a minimum at c. Similarly other cases can be proved.

Example 7.15 In a right circular cone of given volume, if the total surface area is minimum, then prove that sin=1/3, where is the semi-vertical angle.

**Indeterminate Forms: L’Hospital’s Rule**

**Theorem 7.13** Let cR and D=(c-r,c)(c,c+r) for some r>0. Let f,g:DR be differentiable functions such that and . Suppose g(1)(x)0 for all xD and =L. Then =L. Here L can be a real number or .

**Proof** Define F,G :(c-r,c+r)R by F(x)=f(x), xc; F(c)=0 and G(x)=g(x), xc; G(c)=0. Then and . Thus F and G are continuous on (c-r,c+r). Take x(c-r,c+r). F and G satisfy conditions of Cauchy’s MVT between c and x [NOTE: x<c or x>c]. Hence for some d between c and x. As xc, dc. Thus = = .

**Note** If there are standard limits available, then standard limits should be directly used and L’Hospital’s rule should not be used there e.g. =1.

Example 7.16 = =…==0.

Example 7.17 Let = L (R); find a and L.

L==. Since lim(3x2)=0 as x0, =0, that is, a=-2. Now L= may be evaluated using L’Hospital’s Rule.

Example 7.18 Let=1. Prove that a=-5/2,b=-3/2.

Example 7.19 Show that exists but does not exist where f(x)=x+sin x, g(x)=x

**MORE PROBLEMS ON MVTs**

1. Prove that , if a<b. Hence show that .

Applying MVT on f(x)=tan-1x on [a,b], ,a<c<b. Since a<c<b, . For the second part, let b=4/3, a=1.

2. Use Taylor’s finite series with remainder after 3 terms of sin x to evaluate sin 510 and make an estimate of the error made.

Let x=510=, x0=450=. Then x-x0=.

sin 510=sin 450+cos 450 –sin 450.

The absolute value of the error term =< and sin 510=0.777 (correct to 3 decimal places).

3. Prove that between any two real roots of exsin x=1, there is at least one real root of ex cos x=-1.

Let f(x)=e-x-sin x. Let a,b (a<b) be any two real roots of ex sin x=1. Thus f(a)=f(b)=0; other conditions of Rolle’s Theorem can be verified. Thus there exists c, a<c<b, such that f/(c)=0. Thus –e-c-cos c=0. Hence c is a root of ex cos x=-1lying between a and b.

4. Prove that by MVT.

Applying MVT on sin-1x over [.5,.6], we get sin-1(.6)-sin-1(.5)=.1, .5<c<.6. Thus

. Hence the result.

5. Prove that = cot c, a<c<b.

By Cauchy MVT, , a<c<b.

6. 0, for 0.

sin x-sin 0=x cos c,0<c<x. Thus 0.

7. Use MVT to prove .

Applying MVT on f(t)=ln(1+t) on [0,x], we get ln(1+x)-ln 1=(x-0). Since 0<c<x, result follows.

8. If f and g be continuous on [a,b] and derivable on (a,b), then prove that there exists c, a<c<b, such that .

Let F(x)=. Then F/(x)=. F is continuous on [a,b] and F/(x) exists in (a,b). Using MVT, F(b)-F(a)=(b-a)F/(c), a<c<b. Hence the result (note:F(a)=0)

9. A twice differentiable function f on [a,b] is such that f(a)=0=f(b) and f(x0)>0 where a<x0<b. Prove that there exists c, a<c<b, such that f//(c)<0.

f and f/ both exist and are continuous on [a,b]. Applying MVT to f on [a,x0] and [x0,b], we obtain

f(x0)=f(a)+(x0-a)f/(c1), f(b)=f(x0)+(b-x0)f/(c2), a<c1<x0, x0<c2<b. Since f(a)=0=f(b), f/(c1)=, f/(c2)=-, a<c1<x0<c2<b. Applying MVT to f/ on [c1,c2],there exists c(c1,c2) such that f//(c)=. Thus f//(c)=<0.

10. Assuming f//(x) to be continuous on [a,b],prove that f(c)-f(a) f(b)=(c-a)(c-b)f//(d), where c,d(a,b).

Let g(x)=(b-a) f(x)-(b-x)f(a)-(x-a)f(b)-(b-a)(x-a)(x-b)A, where A is a constant to be determined such that g(c)=0. Thus A=. Clearly g(a)=g(b)=0 and g satisfies all conditions of Rolle’s Theorem on each of [a,c] and [c,b]. Hence there exists d1 and d2 in (a,c) and (c,b) respectively such that g/(d1)=0=g/(d2). Again g/(x)=(b-a)f/(x)+f(a)-f(b)-(b-a){2x-(a+b)}A which is continuous on [a,b] and derivable on (a,b) and in particular on [d1,d2]. Also g/(d1)=0=g/(d2). Thus by Rolle’s Theorem, there exists d(d1,d2) such that g//(d)=0. But g//(x)=(b-a)f//(x)-(b-a)2A. Thus f//(d)=2A. Hence A= f//(d); result follows by equating two values of A.

11. Let u,v,u/,v/ be all continuous on R and uv/-u/v0 in R. Prove that between any two consecutive real roots of u=0lies one real root of v=0 and between any two consecutive real roots of v=0 lies one real root of u=0.

Let a,b be any two consecutive real roots of u=0, a<b. since u(a)=0=u(b), v(a)0 and v(b) by given condition. If possible, let v=0 have no real root in (a,b). Then v0 in [a,b]. Let f=

In [a,b]. Then f is continuous on [a,b], f/=exists in (a,b) and f(a)=0=f(b). By Rolle’s Theorem, there exists c(a,b) such that f/(c)=0. This implies uv/-u/v=0 at c, contradiction. Thus v has a real root in (a,b). Similarly other part.

**Note** Taking u(x)=sin x, v(x)= cos x, it follows that between any two consecutive real roots of sin lies a real root of cos.

12. Use MVT to prove 0<, for x>0.

Let a>0. Let f(x)=ex in [0,a]. By MVT, there exists c in (0,a) such that f(a)-f(0)=af/(ca),0<c<1. Thus ea-1=ceca. Thus , that is, 0< =c<1. Result follows by arbitrariness of a>0.

13. If a function f has finite derivative at each point in (a,b) (a,b real, a<b) and if for c(a,b), is finite, say L, prove that f/(c)=L.

Let us choose >0 such that (c,c+(a,b). Since f is differentiable on (a,b),it is differentiable on (c,c+. Let (hn) be a sequence of positive real numbers such that lim(hn)=0. Then there exists natural k such that n k implies 0<hn<. By MVT applied to f on [c,c+hn] for nk, where 0<<1.>0 for nk and lim()=0. Since =L, (=L by sequential criterion. Hence L. Since f has a finite derivative at c, Rf/(c). By sequential criteria, =Rf/(c). Hence Rf/(c)=L. Since f/(c) exists, f/(c)=Rf/(c)=L.

14. Let I be an interval. If f:IR be such that f/ exists and is bounded on I, then f is uniformly continuous on I.

Let x1,x2I with x1<x2. Since f/ is bounded on I, there exists k>0 such that , for all x in I.f satisfies conditions of MVT on [x1,x2] and thus there exists c(x1,x2) such that f/(c)=. Hence , for all x1,x2 in I. Thus f is uniformly continuous on I.

15. f is differentiable on [0,2], f(0)=0, f(1)=2,f(2)=1. Prove that f/(c)=0 for some c in (0,2).

Applying Lagrange’s MVT to f on [0,1] and [1,2], there exists c1 and c2 in (0,1) and (1,2) such that f/(c1)>0 and f/(c2)<0; result follows on applying Darboux’s Theorem to f on [c1,c2].

16. If f is differentiable on [0,1], show by Cauchy MVT that the equation f(1)-f(0)= has at least one solution in (0,1).

Let g:[0,1]R, g(x)=x2. By Cauchy MVT, , for some c in (0,1). Hence f(1)-f(0)=, showing that c(0,1) is a solution to f(1)-f(0)=.

17. A function f is thrice differentiable on [a,b] and f(a)=f(b)=0, f/(a)=f/(b)=0. Prove that f(3)(c)=0 for some c(a,b).

By Rolle’s Theorem applied to f on [a,b] , there exists d(a,b)such that f(1)(d)=0. Applying Rolle’s Theorem to f(1) on [a,d] and [d,b], there exists e(a,d) and g(d,b) such that f(2)(e)=0 and f(2)(g)=0. Applying Rolle’s Theorem to f(2) on [e,g], there exists h(e,g)(a,b) such that f(3)(h)=0.

18. If f(2)(x)0 on [a,b], prove that f for any x1,x2 in [a,b].

Let x1<x2. Applying MVT to f on and , we get f()-f(x1)= f/(c), x1<c< and f(x2)- f= f/(d),<d<x2. Subtracting, 2 f-f(x1)-f(x2)=[f/(c)- f/(d)]=.(d-c)f//(m) (by MVT to f/ on [c,d]) 0. Hence the result.

19. Use Taylor’s Theorem to prove that 1+<, if x>0.

Let f(x)=, x>0. f(1)(x)=, f(3)(x)=.

By Taylor’s Theorem with Lagrange’s form of remainder (after 3 terms)for x>0

=f(x)=f(0)+xf(1)(0)++, for some c in (0,x)

=1+>1+.

Again, by Taylor’s Theorem with Lagrange’s form of remainder (after 2 terms)for x>0

=f(x)=f(0)+xf(1)(0)+ for some d in (0,x)

=1+-<1+. Hence the result.

20. If f(2) is continuous on some neighbourhood of c, prove that f(2)(c).

Let f(2) be continuous on (c-,c+), >0.By Taylor’s Theorem with Lagrange’s form of remainder (after 2 terms)for any h satisfying 0<h<,

f(c+h)=f(c)+hf(1)(c)+ and

f(c-h)=f(c)-hf(1)(c)+, 0<.

Hence f(c+h)+f(c-h)-2f(c)=. Thus . Since f(2) is continuous at c, f(2)(c). Thus f(2)(c).

21. if f(2) is continuous at a and f(2)(a)0, prove that , where is given by f(a+h)=f(a)+hf(1)(a+h), 0<.

Since f(2) is continuous at a, f(2) exists in some nbd. (a-) of a.

**Case 1** Let 0<h<

By Taylor’s Theorem with Lagrange’s form of remainder after 2 terms, f(a+h)=f(a)+hf(1)(a)+, 0<<1.Comparing, f(1)(a+h)= f(1)(a)+.

Applying Lagange’s MVT to f(1) on[a,a+h], we have f(1)(a+h)= f(1)(a)+h, 0<<1. Comparing, . Taking limit as h0+, we have

. Since f(2) is continuous at a,f(2)(a). Thus , since f(2)(a)0.

**Case 2** Let -<h<0. Proceeding similarly, . Combining .

22. If f(x)=sin x, prove that , where is given by f(h)=f(0)+hf(1)(h),0<

sin h=h cos(h),0< Also, f(h)=f(0)+hf(1)(0)+f(2)(0)+f(3)(, 0<<1.

Also f(1)(h)=f(1)(0)+hf(2)(0)+,0<.

Thus sin h=h-cos( and cos(=1-cos(**(3)**

From (1) and (2), cos(h)=1-cos() **(4)**. From (3) and (4), 3. Since

23. If f(2) exists in [a,b] and f(1)(a)=f(1)(b), prove that f, for some c in (a,b).

By MVT applied to f on and on ,

f(=f(a)+f(1)(a)+f(2)(c1), for some c1 in (a,

f(=f(b)-f(1)(b)+f(2)(c2), for some c2 in(,b).

Hence f. If , then by Darboux’s Theorem, there exists c in (c1,c2) such that f(2)(c). Thus f). If , then c= c1 and result follows.

24. If f(2) exists in [a-h,a+h], prove that =f(2)(c), for some c in (a-h,a+h).

By MVT applied to f on [a,a+h] and [a-h,a],

f(a+h)=f(a)+hf(1)(a)+f(2)(c1), for some c1 in (a,a+h)

f(a-h)=f(a)-hf(1)(a)+f(2)(c2),for some c2 in (a-h,a).

Thus f(a+h)+f(a-h)-2f(a)=h2. If , then by Darboux’s Theorem, there exists c in (c1,c2)(a-h,a+h) such that f(2)(c) and result follows. If , then c= c1 and result follows.

25. If a function f is such that its derivative f(1) is continuous on [a,b] and is derivable on (a,b), then show that there exists c in (a,b) such that f(b)=f(a)+(b-a)f(1)(a)+f(2)(c).

f and f(1) are continuous on [a,b] and derivable on (a,b). Consider the function g(x)=f(b)-f(x)-(b-x)f(1)(x)-(b-x)2A, where A is a constant to be determined in such a way that g(a)=g(b) holds. Thus f(b)-f(a)- (b-a)f(1)(a)-(b-a)2A=0 **(1)**.

Now g is continuous on [a,b], derivable on (a,b) and g(a)=g(b). By Rolle’s Theorem, there exists c in (a,b) such that g(1)(c)=0. Thus –(b-c)f(2)(c)+2(b-c)A=g(1)(c)=0. Since bc, A=f(2)(c)/2 **(2)**.

From (1) and (2), result follows.

26. If f(1) and g(1) are continuous and differentiable on [a,b], then show that there exists c in (a,b) such that .

Let h(x)=f(x)+(b-x)f(1)(x)+A{g(x)+(b-x)g(1)(x)}, where A is to be determined such that h(a)=h(b) holds. Thus f(a)+(b-a)f(1)(a)+A{g(a)+(b-a)g(1)(a)}=f(b)+Ag(b). Hence A= **(1)**

If we choose value of A as given by (1),then h satisfies all conditions of Rolle’s Theorem. Hence there exists c in (a,b) such that h(1)(c)=0, that is, f(1)(c)- f(1)(c)+(b-c)f(2)(c)+A{g(1)(c)-g(1)(c)+(b-c)g(2)(c)}=0 **(2)**. Equating values of A obtained from (1) and (2), we get the result.

27. If f(1) and g(1) exist on [a,b] and if g(1) does not vanish on (a,b), then prove that there exists c in (a,b) such that .

Let h(x)=f(x)g(x)-g(x)f(a)-f(x)g(b) on [a,b]. Then h is continuous on [a,b], differentiable on (a,b) and h(a)=- f(a)g(b)=h(b). By Rolle’s Theorem applied to h on [a,b], there exists c in (a,b) such that h(1)(c)=0. Thus f(1)(c)g(c)+f(c)g(1)(c)-g(1)(c)f(a)-f(1)(c)g(b)=0. Hence the result.

28. Applying Lagrange’s MVT to f(x)=tan-1x , x real , show that both f and f(1)are uniformly continuous on R.

f(x)=tan-1x and f(1)(x)= are continuous on [a,b] and differentiable on (a,b), for all real a,b.

By MVT applied to f and f(1) on [a,b], tan-1b-tan-1a=(b-a) and for some c,d in (a,b). Thus =< and < [2dd2+1<(d2+1)2 so that]. Hence the result.

29. Find the values of p and q such that .

=

Since lim(3x2)=0 as x0, for the above limit to exist finitely, ; thus 1-p+q=0 **(1)**. Again using L’Hospital’s rule, given limit is equal to

= (given)**(2)**. Solving (1) and (2), p=1/2, q=-1/2.

30. Suppose f is a function such that f(x)>0 for all real x and f(1) is continuous for all real x. If f(1)(t) for all real t, then show that for all x1.

By hypothesis, g(x)= is continuous and derivable for all x1. By MVT, there exists c in (1,x) such that g(x)-g(1)=(x-1)g/(c). Thus as f/(t), for all real t. Hence the result.

31. Let f:[1,3]R be a continuous function that is differentiable in 91,3) with derivative f/(x)=for all x in (1,3). State , with reasons, whether f(3)-f(1)=5 is true or false.

By MVT, there exists c in (1,3) such that f(3)-f(1)=(3-1)f/(c). Thus , absurd.

32. If f(2) exists in [a,b] and for some c in (a,b), show that there exists at least one point c in (a,b) for which f(2)(c)=0.

Applying MVT to f on [a,c] and [c,b], there exists d1,d2 in (a,c) and (c,b) respectively, such that

=f(1)(d1) and = f(1)(d2). Since f(2) exists in [a,b], f(1) is derivable in [d1,d2]. Thus f(1) satisfies conditions of Rolle’s Theorem in [d1,d2]. Thus there exists d in (d1,d2)(a,b) for which f(2)(d)=0.

33. Show that < for 0<x<.

Let f:[0,]R be defined by f(x)= for 0<x and f(0)=1. So f is continuous in [0, f(1)(x)=, 0<x<. Let g(x)=x cos x-sin x in [0,]. g/(x)=- x sin x<0 for 0<x<. So x>0 implies g(x)<g(0)=0. Thus f(1)(x)<0 for 0<x<. Hence f(0)>f(x)>f() for 0<x<. Hence the result.

34. Let f(2) exists in [o,a], a>0. If f(0)=0 and 0<xa, show that there exist c(0,x) such that f(1)(x)-.

We construct the function g:[0,a]R : g(p)=-f(p)+pf(1)(p)+ p2A, where A is to be determined such that g(0)=g(a) holds. As f and f(1) are continuous in [0,a], g is continuous on [0,a]. g(1)(p)=pf(2)(p)+Ap exists in [0,a]. g(0)=g(a) by construction .So g satisfies all conditions of Rolle’s Theorem in [0,a]. Thus there exists c in (0,a) such that g(1)(c)=0. Hence A=-. g(0)=g(a)implies 0=-f(a)+a f(1)(a)+A=-f(a) +a f(1)(a)-. If 0<xa, -f(x)+xf(1)(x)-=0 for some c in (0,x). Thus f(1)(x)-.

35. Let f,g:[a,b]R be such that both are continuous at the end points a and b and they vanish at a and b. Let f(2), g(2) exist in (a,b), g(2)(x)0 in (a,b). If a<c<b and g(c)0, show that there exists d in (a,b) such that .

We construct F:[a,b]R , F(x)=f(c)g(x)-g(c)f(x).By hypothesis, F is continuous on [a,b] and F(1)exists on (a,b). Also F(a)=0=F(b)=F(c). Applying Rolle’s Theorem to F on [a,c] and [c,b] respectively, there exist d1, d2 in (a,c) and (c,b) respectively such that F(1)(d1)=0=F(1)(d2). By hypothesis F(1) is continuous on [d1,d2], F(2) exists in (d1,d2) and F(1)(d1)=0=F(1)(d2). Applying Rolle’s Theorem to F on [d1,d2], there exists d in (d1,d2)(a,b) such that F(2)(d)=0. Thus f(c)g(2)(d)-f(2)(d)g(c)=0. Hence the result.

**FEW MORE EXERCISES**

1. Let s=(. Find a sequence (sn) of points in l2 such that each sn is distinct from s and such that (sn) converges to s in l2.
2. Let f:R2R be defined by f(x,y)=x. Show that f is continuous on R2. Let g:R2R2 be defined by f(x1,x2)=(x2,x1). Show that f is continuous on R2. If f:RR and g:RR are continuous on R, prove that h:R2R2, h(x,y)=(f(x),g(y)) is continuous on R2.
3. Define f:l2l2 by f(x1,x2,…)=(0,x1,x2,…). Prove that f is continuous on l2.
4. If f:RR is continuous on R, prove that f-1(R+) is open in R, where R+ stands for set of all positive real numbers.
5. Let A be the set of all sequences (xn) in l2 such that <1. Prove that A is an open set in l2.
6. If A and B are open(closed) subsets of R, prove that AXB is an open(closed respectively) subset of R2.
7. Give an example of a countable subset of l2 which is dense in l2.
8. Let X be a MS, A⊆B⊆X. If A is dense in B and B is dense in X, prove that A is dense in X.
9. Prove that every bounded subset of R2 is totally bounded.
10. Give an example of a bounded subset of which is not totally bounded.
11. Prove that if A and B are compact subsets of R, then A X B is compact subset of R2.
12. Give an example of a closed bounded subset of l2 which is not compact.

