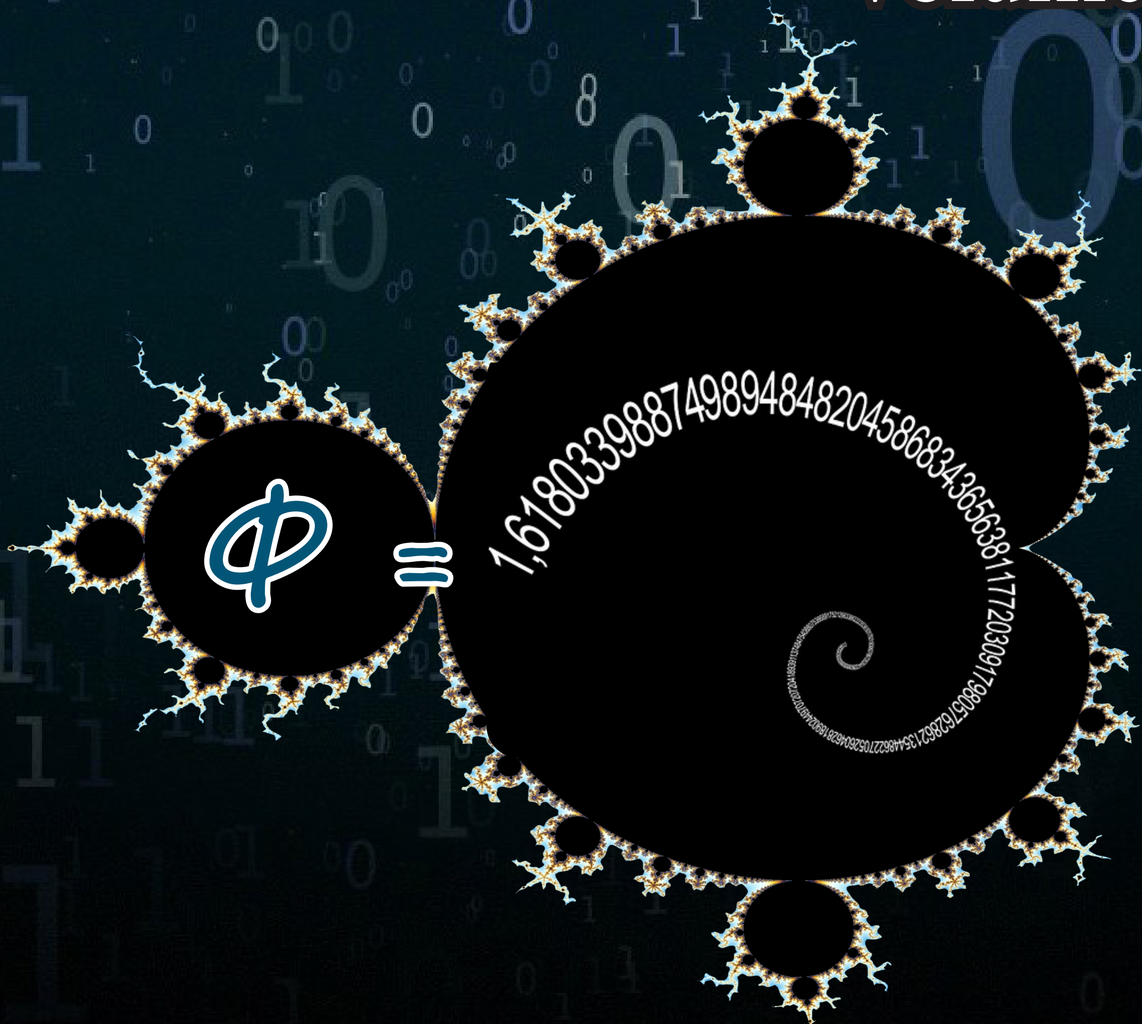
St. Xavier's College (Autonomous), Kolkata

Department of Mathematics

Beacon

Volume IX



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Volume IX

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Department of Mathematics

Message from the
Principal

Rev. Dr. Dominic Savio, S.J.
Principal,
St. Xavier's College (Autonomous), Kolkata.

"It gives me pleasure to note that the Department of Mathematics of our college is bringing out its 9th Edition of the annual departmental magazine, BEACON.

I am glad that by virtue of this publication, the department aims to exhibit the various aspects of the students and showcase the extraordinary cum unique skills and creativity of the students of the department. The magazine has always endeavoured to provide an opportunity to the students to delve deep into the realm of research and development apart from their curriculum.

I would like to extend my heartiest congratulations to the entire department, its faculty members, and the editorial board on their untiring efforts on making the 9th edition of this magazine a success. I wish them success in their efforts in publishing the BEACON.

All the very best.
May God bless you all! Nihil Ultra!"

PRINCIPAL

Message from the
Vice-Principal

Prof. Bertram Da'Silva,
Vice-Principal,
Department of Arts and Science,
St. Xavier's College (Autonomous), Kolkata.

"I am extremely elated to take note that the Department of Mathematics is geared up to publish the 9th edition of their Departmental Magazine, BEACON 2021.

It is through this annual magazine BEACON that the student's creativity and imagination are portrayed every year. We believe in letting the students bloom and grow in whichever field they are good at or are interested in. Thus, we try to encourage all the talents and resources hidden in them. It gives them a touch of the enormous world of science or literature when they traverse various paths to write on different topics.

These articles and works echo many rich frames of minds of the students that the department imbibes in them. At the same juncture, the magazine serves as a realm for students from all other departments to publish articles, art, and literary works.

I would like to congratulate the efforts of the entire department and the editorial team, which resulted in the accomplishment of the yearly chronicle, and I wish them success for their future endeavours. May they keep illuminating the paths for the newcomers to unveil the unknown."

VICE-PRINCIPAL

Message from the
Dean of Science

Dr.Tapati Dutta,
Dean of Science,
St. Xavier's College (Autonomous), Kolkata.

"I am immensely pleased to pen down the message for the annual publication of the Department of Mathematics, BEACON 2021. The publication has always been a platform for the students to showcase and exhibit their plethora of talent in their field of study.

I am elated that by virtue of this magazine, the department targets to mirror the various talents of the students and witnesses the skills and creativities of the students. The magazine has always served as a platform to provide a stage to the students to expound their thoughts through an article of research and their own perspectives.

I would like to extend my heartiest congratulations to the entire department, for their efforts on making this edition of the magazine a success and in their efforts in rolling out the BEACON."

DEAN OF SCIENCE

Message from the

Head of the Department

Prof. Sucharita Roy
Head of the Department,
Department of Mathematics,
St. Xavier's College (Autonomous), Kolkata.

"BEACON is a collaborative effort of the students as well as the professors which include illuminating articles and problems in the subject, poetry, puzzles and many more facets. 'Beacon' is not just a departmental publication as it instigates the students to contribute new ideas, innovations, research, and applications outside classroom topics. We have been able to mirror the teamwork of this department in the magazine, which would not have been possible without the untiring and dogged determination of the students amidst the covid crisis.

I would like to extend my heartfelt gratitude to Father Principal, Vice-Principal, and the Deans for their consistence. I am thankful to the Programme and Publication Committee, for their support. I feel proud to commend and acclaim the Student Editorial board for their hard work to make this issue a reality.

This magazine serves as the epitome for the young brains towards becoming great mathematicians. Last but not the least; I would like to thank all the students who gave tireless effort to bring the 9th edition of BEACON."

Sucharita Roy
HEAD OF THE DEPARTMENT

Message from the

Editor

Prof.Gaurab Tripathi
Assistant Professor,
Department of Mathematics,
St. Xavier's College (Autonomous), Kolkata.

Just like the previous years, the Departmental Magazine "BEACON" has been designed and conceptualised by the combined effort of the students and teachers for this year as well. I feel extremely happy that the responsibility of writing editorial was bestowed upon me, especially amidst the current pandemic situation. Magazine is means to provide platform for students to come forward, identify their talent, discover their potential, and move on the path of progress. It is supposed to garner diverse thoughts and expressions. The aim of publishing 'BEACON' is to encourage creativity of thoughts among the students so that they may learn and grow in every aspect. In this publication, wide varieties of articles are contributed by the students, and I am delighted to see that inventive capacity of our students have been transformed into a tangible way.

I would like to extend my thanks to the editorial team and the students for their untiring efforts in bringing the publication a success.

Gaurab Tripathi

EDITOR

Message from the

Student's Desk

It gives us immense pleasure to bring out the 9th Edition of the Departmental Magazine, Beacon. This chronicle would allow the readers to gauge the principles, values, and ideology of the department. It also serves as a forum for the expression of the literary, artistic, and research-oriented works of the students. The diversity, variety and creativity of the articles and works represent the storehouse of talents present in the department. Each edition of Beacon is a milestone which chronicles our growth and celebrates the achievements of the students. Every line in the magazine reflects a mind where inspiring thoughts take shape.

Regardless of diverse situations, disparate temperaments and assorted backgrounds of the students, the spirit of unanimity, coherence and affinity can be seen in various pages. This edition marks our rise, unravels our artistry, and accords life to our thoughts and aspirations.

Firstly, we would like to thank Prof.Gaurab Tripathi, who supported us from the commencement of this journey till the very end. We are also extremely grateful to Prof. Sucharita Roy, Head of the Department of Mathematics for her constant support and guidance in the accomplishment of this magazine. We convey our gratitude towards the Principal, the Vice-principal and the Dean of Science for their cordial encouragement and consent.

Last but not the least, we would like to extend our heartiest congratulations to the Editorial Board for completing this Herculean task diligently and compiling a collection of articles for the students. We also appreciate the hard work done by the Graphics Committee.

Happy Reading!

Sayantan Porel
(Student Editor)

Sayantan Porel.

Sampurna Mondal
(Associate Student Editor)

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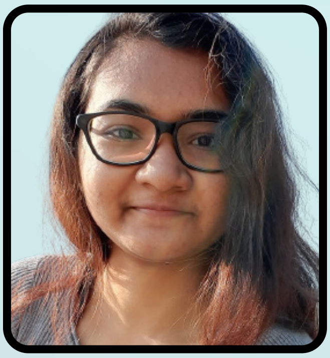

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Mathematical Ideas

Column of Alumnus

RANDOM GRAPHS, SOCIAL NETWORKS AND MATHEMATICS AROUND IT

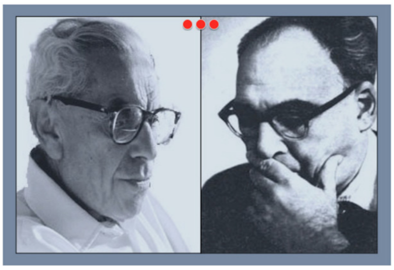
Dr. Rajat Subhra Hazra

Associate Professor
Theoretical Statistics & Mathematics Unit
Indiam Statistical Institute, Kolkata
Shanti Swarup Bhatnagar Awardee,
Science & Technology, 2020

Real-world networks. You might have heard of the term *real-world networks*. This term is used to refer to networks that appear in nature and society. The largest example of a network is our brain. The neurons form for a huge network in the human cortex that is estimated to be of size 10^{10}. The connection between two neurons is through synapses. Another example is cyberspace. Jim Gray was a computer scientist. He won the Turning award in 1998 and said "the emergence of cyberspace and the World Wide Web is like the discovery of a new continent". The vertices of the World Wide Web (WWW) are Web pages and the edges are hyperlinks (URLs) that point from one page to another. In 2011, Google reported that WWW has more than a trillion edges. A further important example is (online) social networks. Each vertex represents a human. Each edge corresponds to some sort of interaction, e.g. friendship. Edges can be either undirected or directed. For instance, an undirected edge occurs if two Facebook accounts are connected to each other, while a directed edge occurs if one account follows the other account in Twitter. Nowadays, social media are part of our daily lives, and can affect society immensely. There are many more examples of networks that effect our day to day lives.

Random graphs. Mathematically, a network is a graph G that is represented by a pair (V, E), where V is the set of vertices and E is the set of edges, i.e., connections between the vertices. Typically, the vertex set is a finite set consisting of n points, i.e., we identify V with $\{1, \ldots, n\}$ and denote an edge between vertices i and j by $\{i, j\}$. The term random graph is used when there is uncertainty involved in the way $\{i, j\}$ is formed. Let us consider an example to explain this. Say, 30 students are selected from a school. These students can be thought of as the vertex set. Every pair of students (*How many are there?*) is given a coin and is asked to toss. If the coin lands on head, then there is friendship between the pair, while if it lands on tail, then there is no friendship. All the coin tosses are independent, in the sense that they do not influence each other. *How many graphs can you form?* A little thought will tell you that the number is 2^{435}. Since the result of the coin toss is uncertain, the formation of the edge is uncertain, and graphs obtained in this way are called random graphs. The best way to describe uncertain objects is through the language of probability.

A simple model. The above experiment was formalised as a model in 1959 by two Hungarian mathematicians: Paul Erdős and Alfred Rényi (see picture).



Although their model is simple to describe, it has many interesting features, which mathematicians are trying to understand. Firstly, the coin we need to use may be biased: the probability of head is a number $p \in [0, 1]$ and the probability of tail is $q = 1 - p$. Instead of 30, they started with a vertex set V of n points, denoted by $\{1, \ldots, n\}$. For any pair of vertices i and j (with $i \neq j$), toss a coin and assign an edge if it lands on head and do not assign an edge if it lands on tail. Perform the tosses independently of each other. The resulting random graph $\mathbf{G}(n, p)$ depends on the two parameters n and p.

One of the important objects to study is the degree of a vertex i, which represents the number of friends of the vertex i. We denote this number by $\deg(i)$. It is an easy exercise to show that

$$\mathbf{P}(\deg(i) = k) = \binom{n-1}{k} p^k (1-p)^{n-1-k}, \qquad k = 0, \ldots, n-1.$$

Think about it! From this it is not hard to see that the average degree is given by $(n-1)p$. So, in the experiment we described before, with 30 students, any student has an average of 14.5 friends. Let us denote the degrees of the vertices by $(\deg(1), \deg(2), \cdots, \deg(n))$. It might interest you to compute and see the following statistical quantities for this basic model.

- Average of the degrees: $\mathbf{E}[\deg(i)] := \sum_{k=1}^{\infty} k\mathbf{P}(\deg(i) = k)$.
- Variance of the degrees: $\mathbf{Var}(\deg(i)) = \mathbf{E}[\deg(i)^2] - \mathbf{E}[\deg(i)]^2$.
- Covariance of the degrees:
  $\mathrm{Cov}(\deg(i), \deg(j)) = \mathbf{E}\left[(\deg(i) - \mathbf{E}[\deg(i)])(\deg(j) - \mathbf{E}[\deg(j)]\right]$.
- **Friendship Paradox:** When you view yourself as the random individual, then show that on an average, a random friend of yours has more friends than you do! *Can you formulate this mathematically?*

Note that $p = 0$ and $p = 1$ are the uninteresting cases and we will avoid them. We claimed that $\mathbf{E}[\deg(i)] = (n-1)p$ (which you may verify). Hence, for fixed p, if we let n be very large, then the average degree is very large for each vertex. This is not a realistic scenario always, and instead we may take $p = c/(n-1)$ for some $c \in (0, \infty)$, for which the average degree equals c for all n. This setting is called *sparse* random graph.

It is important to understand how the degrees behave for large n. One of your tasks is to show that, for $p = c/(n-1)$,

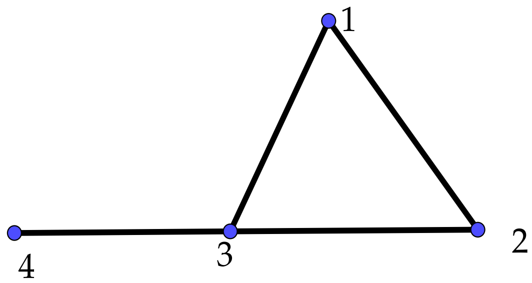
$$\mathbf{P}(\deg(i) = k) \overset{n\to\infty}{\longrightarrow} \frac{e^{-c}c^k}{k!}, \qquad k = 0, 1, \ldots$$

It turns out that the geometry of the graph for large n depends on whether $c < 1$, $c > 1$ or $c = 1$.

Adjacency matrix Let G be an undirected graph on n vertices. Consider the matrix A_G given by

$$A_G(i,j) = \begin{cases} 1 & \text{if there is an edge between } i \text{ and } j, \\ 0 & \text{otherwise.} \end{cases}$$

This is called the adjacency matrix. Note that knowing the whole adjacency matrix gives you full knowledge about the graph itself. Here is an example with 4 vertices:



The corresponding A_G is the 4×4 matrix given by

$$A_G = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We are interested in studying the adjacency matrix for $\mathbf{G}(n, p)$. There are various properties of this matrix that we can look at, for example, eigenvalues, eigenvectors, determinants, invertibility, etc. Recall that an $n \times n$ matrix A is invertible when there exists a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I_n$, where I_n is the identity matrix given by

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

We want to describe an open problem that might interest you! Look at

$$p_n(p) = \mathbf{P}(A_{\mathbf{G}(n,p)} \text{ is invertible})?$$

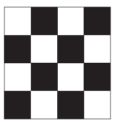
Note that when $p = 0$, the matrix has only zero entries and hence it is not invertible. Also when $p = 1$ it is not invertible (*why?*). The interesting case is $p \in (0, 1)$, where p may depend on n. You can first study the case $p = \frac{1}{2}$ (the problem with a fair coin), and for small values of n compute this probability by hand. Next, can you show that $p_n(\frac{1}{2}) \to 0$ as n becomes large? It would be interesting to do some simulations to estimate $p_n(\frac{1}{2})$. One reason for non-invertibility is the presence of a zero row or a zero column or presence of two rows which are same (In terms of friendships, this amounts to finding out what is the probability that there is a student who has no friends at all!) This should give a lower estimate (*can you guess this estimate?*) on $p_n(p)$, but can it also be used to get an upper bound? The conjecture is

$$2^n p_n(\frac{1}{2}) \to 1.$$

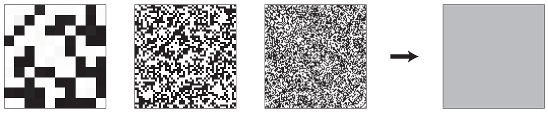
Local view. Another question that matters in social networks these days is: "What does a random graph look like?" This is not a well-posed mathematical question. We have to frame it in a more mathematical format. For example, it is well-known that for $p = c/(n-1)$ the graph locally looks like a tree (*a tree is a graph that does not have any cycle*). There are different ways to make this precise by looking at neighbourhoods of points. For p fixed there are more challenges. One of the useful quantities that have come into the foray is the theory of *graphons*.

This theory was mainly developed by László Lovász (see picture), who was awarded the prestigious Abel Prize this year, a prize that is considered to be the Nobel prize in mathematics. Graphons are functions that arise as limits of adjacency matrices. They are symmetric functions on the unit square $[0, 1]^2$. From the adjacency matrix of a labeled graph, construct the graph's pixel picture by turning the 1's into black squares, the 0's into white square, and scaling everything down to the unit square $[0, 1]^2$. Here is an example of how to construct graphons:

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \longrightarrow$$



Suppose you take an Erdős- Rényi random graph with edge probabilities 1/2. *Where does this random graph converge in terms on graphon?* Pixel pictures may be seen to "converge" graphically; those of larger and larger random graphs with edge probability 1/2, regardless of how they are labeled, seem to converge to a gray square, the constant 1/2 function on $[0, 1]^2$. The following picture simulates this convergence graphically.



László Lovász has build up this theory using fundamental mathematical concepts and today the theory of dense graphs is better understood thanks to this mechanism. For further reading one may consult his book on dense graphs which is freely available on his website.

Social network . In 1967, psychologist Stanley Milgram performed the following experiment. He sent 60 letters to various people in Wichita, Kansas, USA, who were asked to forward the letter to a specific person in Cambridge, Massachusetts, USA. The participants could only pass the letters (by hand) to personal acquaintances who they thought might be able to reach the target, either directly or via "friends of friends". While 50 people responded to the challenge, only 3 letters (roughly 5%) reached their destination. In later experiments, Milgram managed to increase the success rate to 35%, respec- tively, 95% by pretending that the value of the package was high, and by providing more clues about the recipient, such as his/her occupation. The main conclusion from the work of Milgram was that most people are connected by a chain of at most 6 "friends of friends", and this fact was dubbed with the phrase Six Degrees of Separation. The idea of "close connectedness" was first proposed in 1929 by the Hungarian writer Frigyes Karinthy, in a short story called Chains. Later playwright John Guare popularised the phrase when he chose it as the title for his 1990 play. In this play, Ousa, one of the main characters, says: *"Everybody on this planet is separated only by six other people. Six degrees of separation. Between us and everybody else on this planet. The president of the United States. A gondolier in Venice ... It's not just the big names. It's anyone. A native in the rain forest. (...) An Eskimo. I am bound to everyone on this planet by a trail of six people. It is a profound thought."*

The fact that, on average, people can be reached by a chain of at most 6 intermediaries is rather striking. It implies that any two people in remote areas such as Greenland and the Amazon can be linked by a sequence of on average 6 intermediaries. This makes the phrase *It is a small world we live in!* very appropriate indeed. The idea of Milgram was taken up afresh in 2001, with the added possibilities of the computer era. In 2001, Duncan Watts, a professor at Columbia University, recreated Milgram's experiment using an e-mail message as the "package" that needed to be delivered. Surprisingly, after reviewing the data collected by 48,000 senders and 19 targets in 157 different countries, Watts again found that the average number of intermediaries was 6. The research of Watts and the advent of the computer age have opened up new areas of inquiry related to "Six Degrees of Separation" in diverse areas of network theory, such as electrical power grids, disease transmission, corporate communication, and computer circuitry. To put the idea of a small world into network language, we define the vertices of the social graph to be the inhabitants of the world ($n \approx 7 \times 10^9$), and we draw an edge between two people when they "know each other" just like the Erdős-Rényi random graph. Possibilities are various: it could mean that the two people involved have shaken hands at some point, or meet regularly, or address each other on a first-name basis, etc. The precise choice affects the connectivity of the social graph and hence the conclusions we may draw about its topology. One of the main difficulties with social networks is that they are notoriously hard to measure. Questionaires cannot always be trusted, because people have different ideas about what a certain social relation is. Also, questionaires take time to fill out and to collect. As a result, researchers are interested in examples of social networks that can be more easily measured, for instance, because they are electronic. Examples are e-mail networks, or social networks such as Facebook. For more readings one can visit `https://research.fb.com/blog/2016/02/three-and-a-half-degrees-of-separation/` and see that Facebook has now 3.5 degrees of separation.

A task for you: Find out a proper mathematical formulation of *six degrees of separation*?

Conclusion: The article aims to expose you to some buzz-words in the theory of random graphs and it is no way a survey of the brilliant mathematics that lurks behind it. There are wonderful books and expositions in the internet which you can read. Everyday the social networks is providing us with interesting mathematical problem. Question is *is our mathematics community ready to face this challenge?* The answer lies upon the young graduates of mathematics, statistics, physics and computer science.

In mathematical logic, Russell's paradox (also known as Russell's antinomy), is a set-theoretic paradox discovered by the British philosopher and mathematician Bertrand Russell in 1901.Russell's paradox shows that every set theory that contains an unrestricted comprehension principle leads to contradictions . According to the unrestricted comprehension principle, for any sufficiently well-defined property, there is the set of all and only the objects that have that property. Let R be the set of all sets that are not members of themselves. If R is not a member of itself, then its definition entails that it is a member of itself; if it is a member of itself, then it is not a member of itself, since it is the set of all sets that are not members of themselves. The resulting contradiction is Russell's paradox.

COULD THE GREATEST EVER MATHEMATICAL CONJECTURE BE FALSE ?

Prof. Sourabh Bhattacharya

Associate Professor
Interdisciplinary Statistical Research Unit
Indian Statistical Institute, Kolkata

1 Introduction

The great mathematician George Pólya narrated the following anecdote about another great mathematician David Hilbert: *The thirteenth-century German emperor Frederick Barbarossa, who died while on a crusade, was popularly supposed by Germans to be still alive, asleep in a cave deep in the Kyffhäser Mountains, ready to awake and emerge when Germany needed him. Someone asked Hilbert what he would do if, like Barbarossa, he could be revived after a sleep of several centuries. Hilbert: "I would ask whether anyone had proved the Riemann Hypothesis."*

Indeed, it was Hilbert who, in the Second International Congress of Mathematicians held in Paris in the year 1900, had listed the Riemann Hypothesis (henceforth, RH) as the eighth among 23 most important unsolved mathematics problems. Although listed as eighth, it soon became apparent to Hilbert, as the above story clarifies, that RH was the most important of them. If Hilbert is to awake today, he will find that RH remains unproved even to this day, and will go into slumber once again! Some day, he might wake up again following a disturbing hue and cry, only to discover that RH has not been proved, but disproved. This might cause him to fall asleep, along with a major part of the number theory discipline, for good!

In the year 2000, the Clay Mathematics Institute announced one million dollar prize for the resolution of RH, along with six other open problems, of which only the Poincare conjecture has been solved (by Grigori Perelman, who incidentally, declined the prize). Among all the seven problems, RH towers over the rest and there have been innumerable attempts by mathematicians, amateurs and high-profiles, to prove RH and almost every purported proof has been greatly hyped. Alas! All the proofs so far turned out to be erroneous.

What is this RH that is causing Hilbert to repeatedly fall asleep after revivals and giving other mathematicians sleepless nights? As we shall soon discover, it is a deceptively naive statement, which might seem to be a low-hanging fruit to most mathematicians. Yet, as they attempt to lay their hands on the fruit, the tree hauls it up far above their heads like a cruel banter!

Indeed, RH is simply about locations of the roots of a so-called zeta-function of complex numbers – all non-trivial roots of the zeta function have real part 1/2, is the conjecture made by the legendary mathematician Georg Friedrich Bernhard Riemann. Riemann himself seemed to be an embodiment of his conjecture – simple and shy outwardly but with an inward depth as unfathomable as can be!

2 The conjecture and its importance

Let

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots,$$

where s is any complex number. The above zeta function satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s),$$

from which it follows that $-2, -4, -6, -8, \cdots$, are roots (zeros) of the zeta function. These zeros are called the trivial zeros since their existence is trivially asserted from $\sin\left(\frac{\pi s}{2}\right)$ being 0 in the above functional equation.

As can be proved, there are other zeros on the open strip where the real part of s lies between 0 and 1. This strip, containing the non-trivial zeros, is called the *critical strip*. All the non-trivial zeros are complex numbers. The sub-region where the real part of s is $1/2$ is called the *critical line*. Riemann conjectured, in his 1859 paper "On the Number of Prime Numbers Less Than a Given Quantity", his only paper on number theory, that all the non-trivial zeros lie on the critical line. This is the celebrated RH, of which Riemann himself admitted his failure to prove in his "fleeting vain attempts": *One would, of course, like to have a rigorous proof of this, but I have put aside the search for such a proof after some fleeting vain attempts (einigen flüchtigen vergeblichen Versuchen) because it is not necessary for the immediate objective of my investigation.*

The great importance of RH lies in its intimate connection with the prime numbers that are in the heart of the number theory discipline. There are innumerable important results on prime numbers that are proven under the assumption that RH is true. If the RH eventually turns out to be false, a major portion of number theory will come crashing down. Moreover, the prime numbers have important practical applications in the design of encryption methods for military and civilian use such as banking systems. Proof of RH can enhance the consequences of the methods used in such systems.

3 Relationship between RH and prime numbers

For any positive real value x, let $\pi(x)$ denote the number of primes less than or equal to x, that is the prime counting function. Then the prime number theorem says that $\pi(x) \sim \frac{x}{\log(x)}$, as $x \to \infty$, that is,

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log(x)} = 1.$$

The result, although proved and published independently by de la Vallée Poussin and Hadamard in 1896, it was already conjectured by Gauss in 1792. Further, with $Li(x) = \int_2^x \frac{dt}{\log(t)}$, it can be shown that $Li(x) \sim \frac{x}{\log(x)}$, as $x \to \infty$, and RH has the equivalent re-statement

$$\pi(x) = Li(x) + O\left(\sqrt{x}\log(x)\right), \text{ as } x \to \infty.$$

This re-statement implicitly shows the connection between RH and prime numbers. More explicit relationship can be observed as follows.

For $x > 0$, consider the function

$$J(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \frac{1}{4}\pi(x^{1/4}) + \frac{1}{5}\pi(x^{1/5}) + \cdots .$$

Note that $J(x)$ is not an infinite series, since after a first finite number of terms, the remaining terms become zero. Also consider the so-called Möbius function given by

$$\mu(n) = \begin{cases} -1 & \text{if} \quad n \text{ is a square-free positive integer with an odd number of prime factors;} \\ 0 & \text{if} \quad n \text{ has a squared prime factor;} \\ 1 & \text{if} \quad n \text{ is a square-free positive integer with an even number of prime factors,} \end{cases} \tag{1}$$

where, by square-free integer we mean that the integer is not divisible by any perfect square other than 1. With these new functions $J(\cdot)$ and $\mu(\cdot)$, the prime counting function can be expressed as the following:

$$\pi(x) = \sum_n \frac{\mu(n)}{n} J\left(x^{1/n}\right). \tag{2}$$

Using the $J(\cdot)$ function, Riemann's zeta function admits the following expression

$$\frac{1}{s}\log\zeta(s) = \int_0^\infty J(x)x^{-s-1}dx. \tag{3}$$

Now, by inverting (3), $J(x)$ can be expressed in terms of the zeta function, which can be plugged into (2) to write $\pi(x)$ in terms of Riemann's zeta function. The actual expression of $J(x)$ in terms of the zeta function obtained by inversion of (3) is, in fact, the main result of Riemann's 1859 paper and has the following form:

$$J(x) = Li(x) - \sum_{\rho} Li(x^{\rho}) - \log(2) + \int_{x}^{\infty} \frac{dt}{t(t^2-1)\log(t)}, \tag{4}$$

where the values of ρ are nothing but the non-trivial zeros of the zeta function. Thus although at first glance (4) gives the impression that $J(x)$ does not depend upon ζ, it actually depends upon ζ through its non-trivial zeros.

The above arguments show that the primes and the zeta function are explicitly related and hence are RH and the prime numbers. Remarkably, this relation can be viewed as a bridge between number theory (related to $\pi(x)$) and (complex) analysis and calculus (associated with the zeta function).

4 Quantum connection of RH

The concept of random matrix, that is matrix with elements drawn from probability distributions, is central for studying complex quantum dynamical systems. In particular, Hermitian matrices, where the real and imaginary parts of the elements are modeled independently by standard normal distribution, turned out to provide adequate fit to certain quantum-dynamical systems. Importantly, their eigenvalues were suitable candidates for modeling the energy levels observed in experiments. A further characteristic of the eigenvalues is that they exhibited a tendency to repel each other. Hence, studying the probabilistic distribution of the spacings between the eigenvalues was deemed important by many physicists. Freeman Dyson was one such physicist with reasonable expertise in this area.

In 1972, a chance meeting between the number theorist Hugh Montgomery, then a PhD student who had just submitted his thesis on differences between the non-trivial zeros of Riemann's zeta function and Freeman Dyson, led to a very interesting and unlikely connection between the spacings of the non-trivial zeta zeros and eigenvalues of random Hermitian matrices. Further substantiated by the computational works of Andrew Odlyzko, this gave birth to the Montgomery-Odlyzko Law that the distribution of the spacings between successive non-trivial zeros of the Riemann zeta function (suitably normalized) is statistically identical with the distribution of eigenvalue spacings associated with random Hermitian matrices with real and imaginary parts of the elements modeled by standard normal distribution.

The above "law", however, is not a mathematical proof – it is based on experiments. Indeed, since the zeros of the Riemann zeta function are deterministic, the spacings between them are also deterministic. Hence, associating the spacings with distributions seems to be over the line. Proper formulation would need embedding the zeta zero spacings in some appropriate stochastic process framework associated with infinite-dimensional random Hermitian matrices. Nevertheless, the line of thought leading to the law naturally motivated the following questions. Since the operators represented by such random Hermitian matrices can be used to model certain dynamical systems in quantum physics, can there exist a Riemann operator, an operator whose eigenvalues are exactly the zeros of the zeta function? If so, does it represent some dynamical system that could be created in a physics lab, and could that help to prove RH?

Indeed, in 1986 Michael Berry, a British mathematical physicist had published a paper titled "Riemann's Zeta Function: A Model for Quantum Chaos?" This paper used results and discussions widely circulated at this time to come to the conclusion that a Riemann operator corresponds to a chaotic dynamical system. He further argued that the energy levels of that system must be given by the eigenvalues of the system, which are also the imaginary parts of the zeta zeros. And that the periodic orbits in the chaotic system would correspond to the prime numbers!

Alain Connes, a French mathematician and theoretical physicist, considered an alternative route. Instead of attempting to identify operators that correspond to the zeta zeros, he constructed an operator that yields the zeta zeros as its eigenvalues. Specifically, the energy levels (eigenvalues) in his construction are precisely the Riemann zeta zeros on the critical line! At first read of this description, one might form the impression that RH has been proved by Connes, but dear reader, pause a while and re-think. It is a purely constructive work with the aim to re-create the zeros of the zeta function only the critical line. Indeed, Connes offered no explanation as to why there can not be zeros off the critical line.

5 Bayes meets Riemann

As is well-known, the discipline statistics embodies two paradigms – frequentist and Bayesian. While the former considers the unknown parameter to be fixed, the latter attempts to quantify its uncertainty through "prior distribution", which renders the parameter operationally random. Inference about the parameter is obtained from the conditional distribution of the parameter given observed data, namely, the "posterior distribution", which is usually a normalized product of the prior and the likelihood function of the parameter given the data.

Now, given the wholly probabilistic nature of the Bayesian premise, how can the completely deterministic RH fit in? How is the unique nature of RH relevant to Bayesian statistics anyway? Where is the data, what are the parameters, and what is the prior? At first thought, one would be rendered completely clueless. A chance meeting between Montgomery and Dyson opened up the possibilities of very unlikely connections between RH and quantum physics, through random matrices. Thus, the ingredient of randomness that might be relevant to RH can not be totally ruled out, even though we argued that the quantum physics connection was artificial, probabilistically not rigorous and has remained fruitless. Does the Bayesian paradigm, with its solid and coherent foundation, stand a chance?

To explore the possibilities, let us first consider another equivalent statement of the RH, this time in terms of an infinite series involving the Möbius function (1). Consider the following Dirichlet series for the Möbius function:

$$M(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}. \tag{5}$$

It is known that the series diverges and converges for the real parts of s being less than or equal to 1/2 and greater than or equal to 1, respectively. Convergence of $M(s)$ for all s with real part between 1/2 and 1, is equivalent to RH.

If it is at all possible to characterize convergence and divergence of the infinite series in Bayesian terms, then this would of course characterize RH. Moreover, it the Bayesian characterization could lead to useful inference regarding convergence and divergence of (5), then that could count as substantial evidence, either in favour of, or against RH. In this regard, note that even if for a single real value s between 1/2 and 1, $M(s)$ diverges, then RH must be false.

5.1 The motivation for Bayesian characterization

It is interesting to note that the key idea of Bayesian characterization emerged as a response to a simple curiosity of Professor Sucharita Roy, with regard to infinite series, with no connection whatsoever with RH. Professor Roy, the head of the Department of Mathematics at St. Xavier's College, Kolkata, noted with dismay, perhaps like many other mathematics teachers, that although determination of convergence, divergence or oscillation of infinite series is a much-studied problem in classical mathematics, unfortunately there does not yet seem to exist any universal test that can provide conclusive answers regarding convergence of most infinite series. This issue kept preventing her from answering relevant questions from her students regarding series convergence. Hearing of the powerful Bayesian paradigm from some of her (over-enthusiastic!) colleagues as the panacea to all problems, she was left wondering about answering questions of series convergence by surrendering to the Bayesian power. Her PhD supervisor, a Bayesian, considering this an innocuous banter, did not take it seriously at the first thought. However, importance of the banter dawned on him with an afterthought...

5.2 The Bayesian characterization in a nutshell

The Bayesian characterization approach of Professor Roy attempts to provide conclusive answers to the question of series convergence even where all the existing tests fail. The key philosophy here is to embed the deterministic problem of classical mathematics in a stochastic framework, with a notion of probability of convergence of the series, on which an appropriate prior probability distribution is assigned. The data is composed of the partial sums of the infinite series. Specifically, a recursive Bayesian technique is developed, where posterior distributions of the probability of series convergence at successive stages of the partial sums associated with the infinite series are constructed. Roughly, convergence of the recursive posteriors to 1 or 0 as the number of stages is taken to infinity, is equivalent to convergence or divergence, respectively, of the underlying series!

6 Evidence against RH through Bayesian characterization

To the great surprise of Professor Roy and her supervisor, the Bayesian characterization yielded results that provided strong evidence against RH. Specifically, for real values in the interval $(0.5, 0.72)$, the recursive posterior distributions exhibited convergence to zero, as the number of stages are grown sufficiently large. Professor Roy has also extended the Bayesian theory to encompasses infinite series with finite as well as countably infinite number of limit points. The multiple limit point theory for investigation of RH validity provided identical results of evidence against RH. Although these results are by no means proof that RH is false (as they are based on a quite large, but finite number of recursive stages), they have definitely strengthened our belief that the conjecture cannot be supported.

7 Littlewood's result and the S function: further reasons to question RH

If most of the mathematicians of all ages believe in the truth of RH, this must be due to the sheer weight of evidence gained by numerical computation of the non-trivial zeros of the zeta function.

As unlikely it may sound, the first computations of zeros were performed by Riemann himself, in the context of his 1859 paper. Although Riemann did not bother to publish these computations, they were underneath the basis of his celebrated conjecture. Siegel, through dedicated study of Riemann's notes, was finally able to lay his hands on the ingenious computational method of Riemann, which was published in the 1930s and came to be known as the Riemann-Siegel formula. This was to form the basis of large scale computations of the zeta function.

Since 1859, a very large number of non-trivial zeta zeros have been computed by various researchers around the world. In particular, in the year 2004, X. Gourdon computed 10^{12} non-trivial zeros. In all the cases, the zeros fell comfortably on the critical line. Andrew Odlyzko also computed the highest non-trivial zero of the zeta function known to date, the $10,000,000,000,000,000,010,000$-th, which turned out to be at argument $\frac{1}{2} + 1,370,919,909,931,995,309,568.33539i$, where $i = \sqrt{-1}$.

All the above computational results seem to point towards validity of RH. But that such computational evidences can be rather weak can be illustrated by a conjecture regarding the prime counting function and its approximation, the Li function. All numerical computations had indicated that $Li(N) > \pi(N)$ for all positive integers N, and almost all mathematicians, including both Gauss and Riemann believed in the truth of the conjecture. However, John Edensor Littlewood proved in 1912 that the inequality fails for some N. In 1914 he further proved that the failure occurs for infinitely many N. In 2000 Carter Bays and Richard Hudson showed that failures occur in the vicinity of 1.39822×10^{316}! They further gave some reasons for thinking that these may be the first violations of the conjectured inequality. Indeed, even modern computing power is nowhere near the first violation, and hence no direct computational evidence could have uncovered the truth of the conjecture that Littlewood eventually proved to be false.

It is thus conceivable that to refute RH, we might need computational power that can not be achieved even in very far distant future! To elucidate further, note that the zeta function can be decomposed into two parts, one of which is the so-called S-function. Even when the imaginary part of a root is of the order 10^{23}, S lies between -1 and 1. RH runs the risk of getting into trouble if S rises to more than 100. Since it has been proved by Atle Selberg in 1946 that S is unbounded, eventually S must exceed any given number. Even for S around 100, the imaginary part must be around $10^{10^{10,000}}$! This lies far beyond the reach of any imaginable computing power in even very distant future, to generate a counter-example of RH. Till then, ignorance may be bliss for most of the mathematicians around the world!

8 The non-believers Turing and Littlewood

Among a few great mathematicians who doubted RH were Alan Turing and Littlewood. Surprisingly for Littlewood, it is not his result on refutation of $Li(N) > \pi(N)$, or for that matter, the almost unfathomably large upper bound below which the inequality is violated, a body of research initiated by his own student Samuel Skewes (his upper bound was $e^{e^{e^{79}}}$!), that made him a non-believer. Although Skewes assumed RH to be true for his proof, subsequent researchers, for example, Bays and Hudson, did not assume so. According to Littlewood, "A long-open conjecture in analysis generally turns out to be false. A long-open conjecture in algebra generally turns out to be true." More aptly, it is generally held that Littlewood's inability to prove RH led him to state that it is false (*grapes are sour!*).

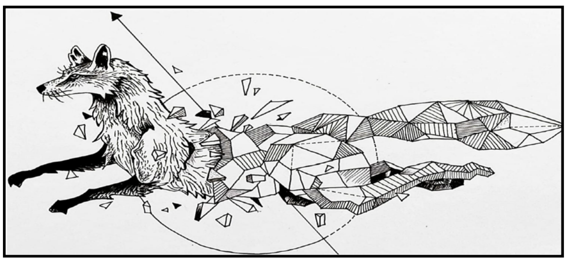
It is Turing's stand that is more interesting. Although it is not known what made him a non-believer, but by the time he was 26 years old (1937), he was convinced that RH was false and set out to construct a counter example by generating a zero off the critical line. For the purpose he conceived the idea of building a mechanical computing device ("zeta function machine"), and applied to the Royal Society for a grant to cover the cost. He himself cut some gear wheels at the engineering department of King's College, Cambridge, where he was lecturing. Unfortunately, in 1939 World War II broke out and Turings's work abruptly came to an end. Given the uncanny insights Turing possessed, there was a fair chance that he could indeed produce a counter example had he got the opportunity!

9 Conclusion

So, what is the chance that RH is correct? We hope that the reader has already begun to believe that it is false... Dear Professor Hilbert, would you also change your mind?

Acknowledgment

We acknowledge the books "Prime Obsession – Berhnard Riemann and the Greatest Unsolved Problem in Mathematics" by John Derbyshire and "The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike" by Peter Borwein, Stephen Choi, Brendan Rooney, and Andrea Weirathmueller, for some historic facts and materials.



PURE MATHEMATICS
vs
COMPUTER PROGRAMMING

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Unity in mathematics has been invaded by various issues like: pure mathematics vs applied mathematics, discrete mathematics vs continuous mathematics, existential vs constructive mathematics. There are numerous reasons behind these, ranging from socio-cultural reasons to availability of financial aids. However, irrespective of the reason, these have some serious consequences. These divisions are bottleneck to the advancement of mathematics, as a whole.

In this article, I intend to point out one such division which I have experienced in my mathematical career so far and explain why this is detrimental to mathematics. I am talking about the cold war between pure mathematics and use of computer programming.

In general, due to the nature of applicability, the applied mathematicians use computer programming and simulation more often than their purist counterparts. However, the shocking part is that there is a taboo about using or learning computer programming among pure mathematicians. I have heard a lot of *pure mathematicians* taking pride in not availing the help of programming languages in their work. Even, Sir Andrew Wiles was quoted to say "I never use a computer". Of course, there is no harm in not using computers for programming. However, having a stigma about it just for the sake of establishing mathematical superiority is pointless. Being able to write a program to test conjectures or just try to see what is happening in a problem is certainly an asset. Let's talk about some concrete examples:

- The notorious prime generating polynomial $f(n) = n^2 + n + 41$ yields prime numbers for $0 \leq n < 40$. However $f(40) = 1681 = 41^2$ is not a prime. Now, suppose you are not aware of this result and have a hunch that $f(n)$ is a prime for all values of n. But, you are not able to prove it. An obvious way is to check the validity of the statement for small values of n. There are two ways to do it, either by hand or by using a programming language. Now, just estimate the amount of time that you can save if you use the later method. One can argue that it may take some more time to do it by hand, but eventually we will reach the same conclusion. But imagine if the first value of n for which the statement fails is large, say $\cong 10^5$. Then in the worst case (without using computer), it may happen that you stick to your hunch that the statement is true and you waste much of your time in trying to prove it.

- Part of the problem to starting math research is that you have no clue what problem to solve. You don't just sit at your desk and wonder, "What theorem should I prove today?" In order to gain a sense for what theorems I want to prove, I always investigate problems on the computer in whatever way I can. Even plotting the curve of a continuous function or testing the validity of a conjecture or theorem using computer programming can provide you a lot of insight about the solution or proof of a problem.

- When it comes to teaching or demonstrating, programming and automation can be used to a large extent e.g., curve plotting, constructing examples and counter examples. One great example of this can be demonstrating uniform convergence of sequence of functions with the help of automation.

This superiority complex among pure mathematicians for not depending upon computers is prevalent, in different extent, almost in every part of the globe. And sadly, they are propagating this among their students. My intention behind writing this article is to request the readers, especially young minds, to come out of this taboo. For, even if Sachin Tendulkar didn't bowl, he would have been equally famous. But his bowling skills have won many matches for India.

> *"Mathematics as an expression of the human mind reflects the active will, the contemplative reason, and the desire for aesthetic perfection. Its basic elements are logic and intuition, analysis and construction, generality and individuality."*
>
> *-Richard Courant*

MATHEMATICS OF STELLAR EVOLUTION : LANE-EMDEN EQUATION

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The evolution of a star is considered to be a quasi-static process, in which the composition changes at a slow rate, thereby, allowing the star to maintain hydro-static equilibrium and thermal equilibrium as well. In this article, we shall see the mathematical modelling of the equilibrium structure of a star of a given composition. The structure of a star is given by the solution obtained by solving a set of differential equations known as the **stellar structure equations**. These differential equations are formulated either in terms of radius (r) or mass (m). Let us derive the differential equations governing stellar evolution.

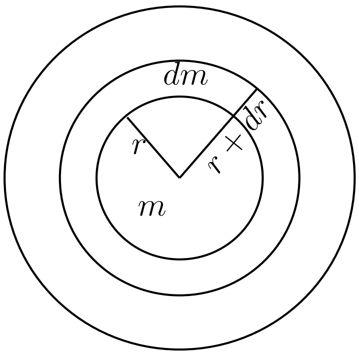


Figure 1: A diagrammatic representation of mass element in a star

The energy equation

Consider a small element of mass dm within a star which is at a constant temperature, density and composition. Assuming spherical symmetry, such an element is chosen as a thin spherical shell of inner radius r and outer radius $r + dr$. So, its volume is given by

$$dV = 4\pi r^2 dr \tag{1}$$

and mass is

$$dm = \rho dV = 4\pi r^2 \rho dr \tag{2}$$

We denote internal energy per unit mass and pressure as u and P respectively. If δQ is the amount of heat absorbed ($\delta Q > 0$) or heat released ($\delta Q < 0$) by the mass element and δW is the work done during the time interval δt, then according to first law of thermodynamics, we get

$$\delta(udm) = dm\delta u = \delta Q + \delta W \tag{3}$$

As due to conservation of mass, dm is constant, we get

$$\delta W = -P\delta dV = -P\delta\left(\frac{dV}{dm}dm\right) = -P\delta\left(\frac{1}{\rho}\right)dm \tag{4}$$

The sources of heat of the mass element are:

(a) nuclear energy release

(b) heat flux balance

Let q be the rate of nuclear energy release per unit mass and $F(m)$ be the heat flow perpendicular to the surface. Then,

$$\delta Q = qdm\delta t + F(m)\delta t - F(m + dm)\delta t \tag{5}$$

But, $F(m + dm) = F(m) + \frac{\partial F}{\partial m}dm$. So,

$$\delta Q = \left(q - \frac{\partial F}{\partial m}\right)dm\delta t \tag{6}$$

Consequently, we get from equation (3)

$$dm\delta u + P\delta\left(\frac{1}{\rho}\right)dm = \left(q - \frac{\partial F}{\partial m}\right)dm\delta t \tag{7}$$

In the limit, $\delta t \longrightarrow 0$, we get

$$\dot{u} + P\dot{\left(\frac{1}{\rho}\right)} = q - \frac{\partial F}{\partial m} \tag{8}$$

In thermal equilibrium, the left hand side of the equation vanishes, and we get

$$\frac{dF}{dm} = q \tag{9}$$

The equation of motion

Consider a small cylindrical volume element within the star, with axis length dr and cross-sectional area dS. Let the denisty within the elemnt be ρ and mass be Δm. Then, we have

$$\Delta m = \rho dr dS \tag{10}$$

The forces acting on the element are

(a) the gravitational force exerted by the sphere interior to r

(b) force due to the gas pressure around the element

The equation of motion is

$$\Delta m \frac{\partial^2 r}{\partial t^2} = -\frac{Gm\Delta m}{r^2} + P(r)dS - P(r+dr)dS \tag{11}$$

But, $P(r+dr) = P(r) + \frac{\partial P}{\partial r}dr$. So,

$$\Delta m \frac{\partial^2 r}{\partial t^2} = -\frac{Gm\Delta m}{r^2} - \frac{\partial P}{\partial r}\frac{\Delta m}{\rho} \tag{12}$$

Dividing by Δm, we get

$$\frac{\partial^2 r}{\partial t^2} = -\frac{Gm}{r^2} - \frac{\partial P}{\partial r}\frac{1}{\rho} \tag{13}$$

In hydrostatic equilibrium, the acceleration is negligible and gravitational force and pressure force balances each other, giving

$$\frac{dP}{dr} = -\rho\frac{Gm}{r^2} \tag{14}$$

Radiative transfer

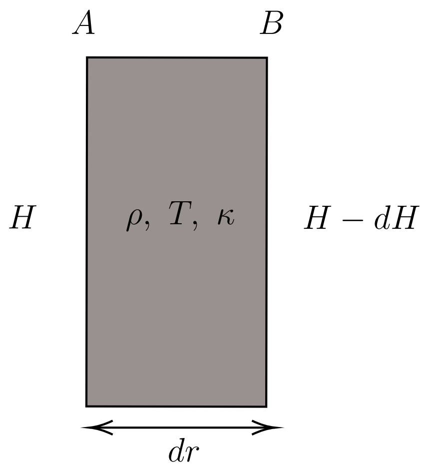


Figure 2: Radiative transfer through a slab

Let us consider a slab of thickness dr and density ρ. Radiation flux H incident at A emerges from B after losing dH amount of flux. The amount of absorbed flux is given by

$$dH = -\kappa H \rho dr \tag{15}$$

where κ is the opacity coefficient. The absorption of radiation energy by the slab involves an amount of momentum. The momentum absorbed in unit time is $\frac{|dH|}{c}$. The increase in momentum is the difference in radiation pressure. So, we have

$$\frac{\kappa H \rho}{c} = -\frac{dP_{rad}}{dr} \tag{16}$$

Radiation pressure is given by

$$P_{rad} = \frac{1}{3}aT^4 \tag{17}$$

where a is the radiation constant. Then

$$\frac{dP_{rad}}{dr} = \frac{4}{c}aT^3\frac{dT}{dr} \tag{18}$$

Therefore,

$$\frac{\kappa H\rho}{c} = -\frac{4}{c}aT^3\frac{dT}{dr} \implies H = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr} \tag{19}$$

The total flux passing the spherical surface of radius r is given by

$$F = 4\pi r^2 H = -4\pi r^2\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr} \tag{20}$$

So, we get

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{F}{4\pi r^2} \tag{21}$$

Hence, the **stellar equations** are either

$$\frac{dP}{dr} = -\rho\frac{Gm}{r^2} \tag{22}$$

$$\frac{dm}{dr} = 4\pi r^2\rho \tag{23}$$

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{F}{4\pi r^2} \tag{24}$$

$$\frac{dF}{dr} = 4\pi r^2\rho q \tag{25}$$

or in the form $(dm = 4\pi r^2 dr)$

$$\frac{dP}{dm} = -\rho\frac{Gm}{4\pi r^4} \tag{26}$$

$$\frac{dr}{dm} = \frac{1}{4\pi r^2\rho} \tag{27}$$

$$\frac{dT}{dm} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{F}{(4\pi r^2)^2} \tag{28}$$

$$\frac{dF}{dm} = q \tag{29}$$

where ρ, P, T, F are density, pressure, temperature and flux density of the star respectively. The constants G, κ and a are universal gravitational constant, opacity coefficient and radiation constant respectively. Here, q is the energy released per unit mass.

The first equation is known as the equation of hydro-static equilibrium, the second is the continuity equation, the third being the radiative transfer equation (provided the main energy transfer takes place by radiative transfer) and the fourth equation is the thermal-equilibrium equation. The above set of differential equations is supplemented by the relation of total pressure

$$P = \frac{\mathcal{R}}{\mu_1}\rho T + P_e + \frac{1}{3}aT^4 \tag{30}$$

and the relation of κ and q as

$$\kappa = \kappa_0 \rho^a T^b \tag{31}$$

$$q = q_0 \rho^m T^n \tag{32}$$

Integration of the differential equations give the distribution of T, ρ, m (or r) and F. The model that I am going to describe here is known as the **polytropic model**. This model is called so because of the form of the equation of state considered here. We take the polytropic equation of state given by

$$P = K\rho^\gamma \tag{33}$$

where K and γ are constants. γ is related to the polytropic index n by the relation $\gamma = 1 + \frac{1}{n}$.

Multiplying the hydrostatic equilibrium equation by $\frac{r^2}{\rho}$ and differentiating with respect to r, we get

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dm}{dr} \tag{34}$$

Substituting equation (23) above, we get

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \tag{35}$$

Replacing P by ρ using the polytropic equation of state, we get

$$\frac{(n+1)K}{4\pi Gn}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{\frac{n-1}{n}}}\frac{d\rho}{dr}\right) = -\rho \tag{36}$$

The solution to the above equation given by $\rho(r)$ with $0 \leq r \leq R$ is called a **polytrope**. To completely determine ρ, we need two boundary conditions, given by $\rho = 0$ at $r = R$ (at surface) and $\frac{d\rho}{dr} = 0$ at $r = 0$ (at centre) due to hydrostatic equilibrium ($\frac{dP}{dr} = 0$).

Let us define a dimensionless variable θ with $0 \leq \theta \leq 1$ as

$$\rho = \rho_c \theta^n \tag{37}$$

where ρ_c is the critical density.

Substituting in equation (36), we get

$$\left[\frac{(n+1)K}{4\pi G\rho_c^{\frac{n-1}{n}}}\right]\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n \tag{38}$$

Let us take $\left[\frac{(n+1)K}{4\pi G\rho_c^{\frac{n-1}{n}}}\right] = \alpha^2$ and replace r by another dimensionless variable ξ as $r = \alpha\xi$, to get

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \tag{39}$$

This equation is known as the famous **Lane-Emden equation** of index n subject to the boundary conditions $\theta = 1$ and $\frac{d\theta}{d\xi} = 0$ at $\xi = 0$. The outer boundary (the surface) is the first location where $\rho = 0$, or equivalently $\theta(\xi) = 0$. That location is called ξ_1. The formal solution may have additional zeros at larger values of ξ, but $\xi > \xi_1$ is not relevant for stellar models. The solution of the Lane-Emden equation for $n = 0$ to $n = 6$ is given in Fig (3).

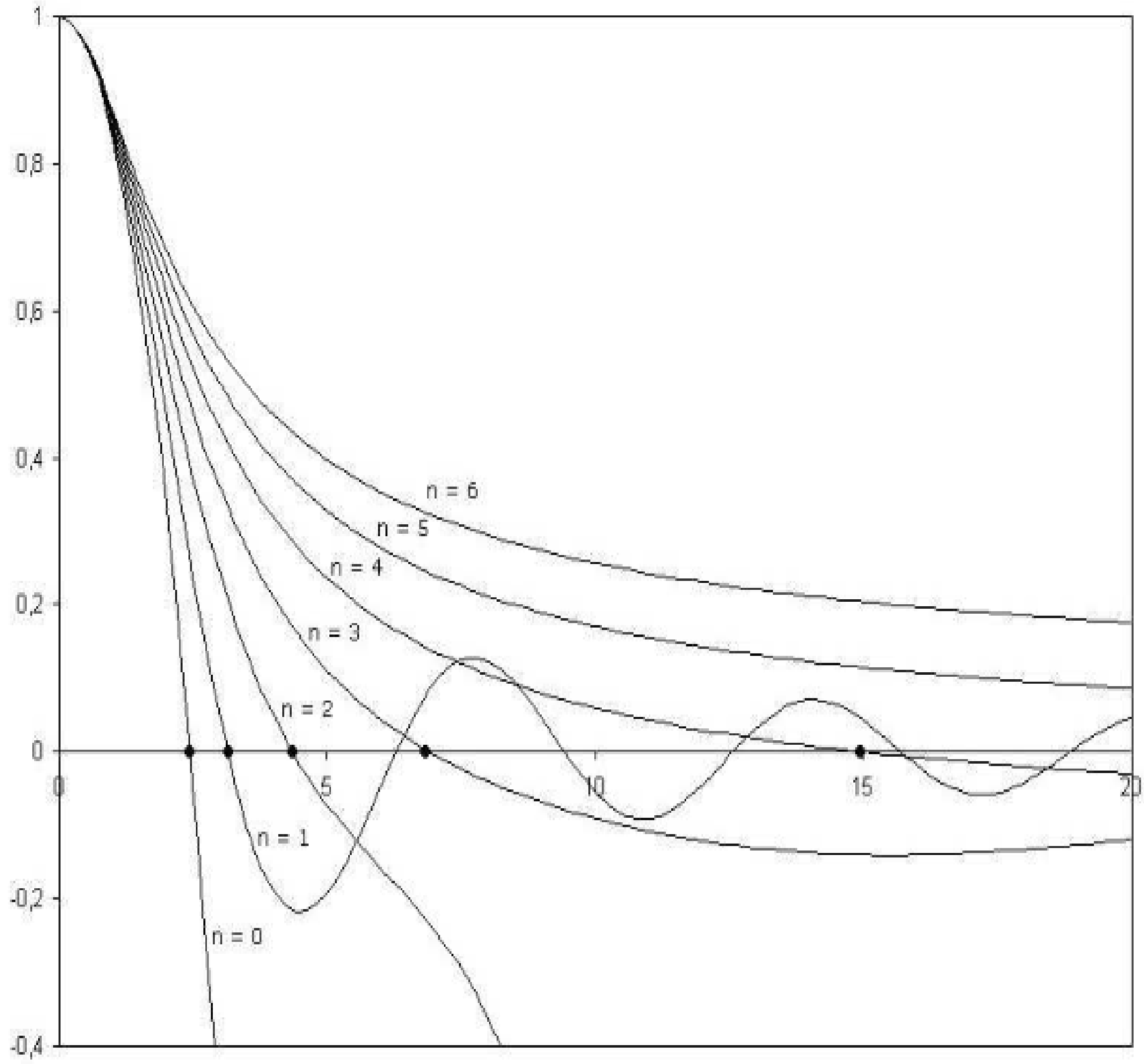
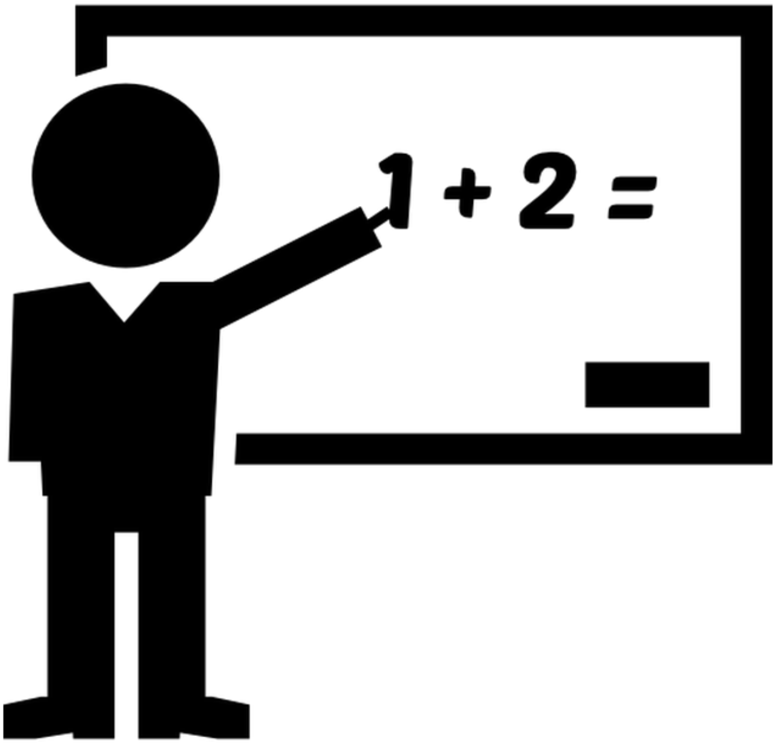


Figure 3: Solutions θ to the Lane-Emden equation for various values of n. The black dots represent the first zeros which corresponds to the stellar radius.

For $n = 0$, the density of the solution as a function of radius is constant, $\rho(r) = \rho^c$. This is the solution for a constant density incompressible sphere. $n = 1$ to 1.5 approximates a fully convective star, i.e. a very cool late-type star such as a white dwarf. For $n = 3$ there is no analytical solution but we have an approximated solution known as the **Eddington approximation** which is useful as it corresponds to a fully radiative star, which is a useful approximation for the Sun. White dwarfs and neutron stars can be approximated as fully degenerate stars, with the lower mass white dwarfs being approximated as a case of non-relativistic degeneracy, and the higher mass white dwarfs and all neutron stars are cases of fully relativistic degeneracy. We thus can represent their internal structure by polytropes with $n = 1.5$ for the non-relativistic case, and $n = 3$ for the relativistic case. For $n > 5$, the binding energy is positive, and hence such a polytrope cannot represent a real star.

False positive paradox describes situations where there are more false positive test results than true positives. For example, 50 of 1,000 people test positive for an infection, but only 10 have the infection, meaning 40 tests were false positives. The probability of a positive test result is determined not only by the accuracy of the test but also by the characteristics of the sampled population. When the prevalence, the proportion of those who have a given condition, is lower than the test's false positive rate, even tests that have a very low chance of giving a false positive in an individual case will give more false than true positives overall. The paradox surprises most people.It is especially counter-intuitive when interpreting a positive result in a test on a low-prevalence population after having dealt with positive results drawn from a high-prevalence population. If the false positive rate of the test is higher than the

Column of Professor



ROMANTIC LOVE - A MATHEMATICAL DISCUSSION

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Love is the customary word used in literature. Most of the songs and poetries of Tagore are based on love. In cinematography too, most of the stories are based on love especially romantic love viz. DDLJ, Devdas, Sat Panke Bandha, Roman holiday, Titanic, etc is an unending list for this. Students in the age group of 15-23 discuss mostly about love and love life. So, we all know it is a set of emotions, but the question is that whether we can explain it from mathematical point of view? Strogatz presented a brief discussion on love affairs and several related mathematical exercises (1994) essentially the same model was described earlier by Rapport (1960) and studied by Radzicki (1993), and several other researchers studied this including different complexity. Here we will discuss the simplest linear model and its interpretation and try to confirm the same behaviour from our experience.

As we all know that the most popular lovers are Romeo and Juliet. Here also we are using the same names.

Let, R(t) = Romeo's love / hate for Juliet at time 't', and

J(t) = Juliet's love / hate for Romeo at time 't' (where, positive value of R and J signify Love and negative value signifies Hate)

The rate of increase of their love is given by

$$\frac{dR}{dt} = aR + bJ \;\; \& \;\; \frac{dJ}{dt} = cR + dJ$$

Here 'a' and 'b' characterizes Romeo's romantic style and 'c', and 'd' characterizes the same of Juliet. The parameter 'a' describes the extent to which Romeo is encouraged by his own feeling and 'b' denotes the extent to which he is encouraged by Juliet's feeling. The parameters 'c' and 'd' have the equivalent significances from the perspective of Juliet. These four parameters a, b, c, d can be positive or negative. We can write the expression as:

$$\frac{d}{dt}\begin{pmatrix} R \\ J \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} R \\ J \end{pmatrix}$$

Hence, if we consider $X = \begin{pmatrix} R \\ J \end{pmatrix}, A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then equation can be written as:

$\dot{X} = AX$....... (1)

Types of Lovers and various cases:

From the perspective of Romeo four different styles of romance can be exhibited by him which are given below:

(i) Eager beaver: $a > 0, b > 0$
(ii) Narcissistic nerd: $a > 0, b < 0$
(iii) Cautious (or, secure) lover: $a < 0, b > 0$
(iv) Hermit: $a < 0, b < 0$.

The same format of styles can be generated from Juliet involving the parameters 'c' and 'd' respectively. Later it has been discussed in detail about what happens in the 4 different cases when various kinds of lovers meet each other and what the outcome is. We have taken 6 instances for all the respective cases and tried to reach to a conclusion.

The six instances are:

I. Both love each other $R, J > 0$.
II. Both hate each other $R, J < 0$.
III. Romeo loves, Juliet hates and love > hate i.e., $|R|>|J|$
IV. Romeo hates, Juliet loves and love > hate i.e., $|R|<|J|$
V. Romeo loves, Juliet hates and love < hate i.e., $|R|<|J|$
VI. Romeo hates, Juliet loves and love < hate i.e. $|R|>|J|$

So, we reached at two-dimensional Linear Dynamical system. Before going to discuss further we find the solution of equation (1) qualitatively.

Solution of AX = 0 is called critical point (CP) or equilibrium point. So, if X* is CP then AX*=0 i.e. at X=X* we have $\dot{X}$=0. So, it will not move if the particle is at CP. Now we will solve Equation (1) by eigenvalue eigenvector method. We can find eigenvalues by solving characteristic equation of A, i.e., $|A - \lambda I| = 0$.

The characteristic equation |A- λI|=0 is given by:

λ^2–(a+d) λ+(ad-bc)=0,

Or, λ^2 -T λ+Δ=0 ,where T=Tr(A)=a+d & Δ=|A|=ad-bc.

So, eigenvalues are given by $\lambda_{1,2} = \frac{T \pm \sqrt{T^2 - 4\Delta}}{2}$

Case(1): $T^2 > 4\Delta$. Eigenvaues are real and ditinct .

If λ_1, λ_2 *are eigenvalues and* $\overrightarrow{v_1}, \overrightarrow{v_2}$ *are eigenvectors given by* $A\vec{v} = \lambda\vec{v}, \vec{v} \neq 0$.

Then solution of equation (1) is given by $X(t) = c_1\overrightarrow{v_1}e^{\lambda_1 t} + c_2\overrightarrow{v_2}e^{\lambda_2 t}$ for some constant c_1 *and* c_2.

Case(2): $T^2 < 4\Delta$. Eigenvaues are complex conjugate.

If $\lambda_{1,2} = \lambda \pm i\mu$ *are eigenvalues and* $\vec{v} = \vec{u} \pm i\vec{w}$ are respective eigenvector, then X(t) is a linear combination of X_1(t) and X_2(t) ,where

$X_1(t) = e^{\lambda t}(\vec{u}\cos\mu t - \vec{w}\sin\mu t)$ & $X_2(t) = e^{\lambda t}(\vec{u}\sin\mu t + \vec{w}\cos\mu t)$

Case(3): $T^2 = 4\Delta$. Eigenvaues are real and equal . Here two subcases will arise viz. geometric multiplicity may be 1 or 2. For 2 we will take it like case (1) but for 1 problem arises since we get only one eigenvector. In this case we can use generalised eigenvector to get its Jordan canonical form. (We will discuss this case through an example)

We are dicussing the cases through example. consider $X = \begin{pmatrix} R \\ J \end{pmatrix}$ and $\dot{X} = AX$

Let $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Here actually we are getting decoupled equation.

$\dot{R} = \lambda_1 R$ *and* $\dot{J} = \lambda_2 J$ which can be solved separately but we use general method. Clearly eigenvalues are diagonal elements λ_1, λ_2 *and eigenvectors are* $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ *respectly*.So solution is $X = c_1 e^{\lambda_1 t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda_2 t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. So $R = c_1 e^{\lambda_1 t}$ *and* $J = c_2 e^{\lambda_2 t}$

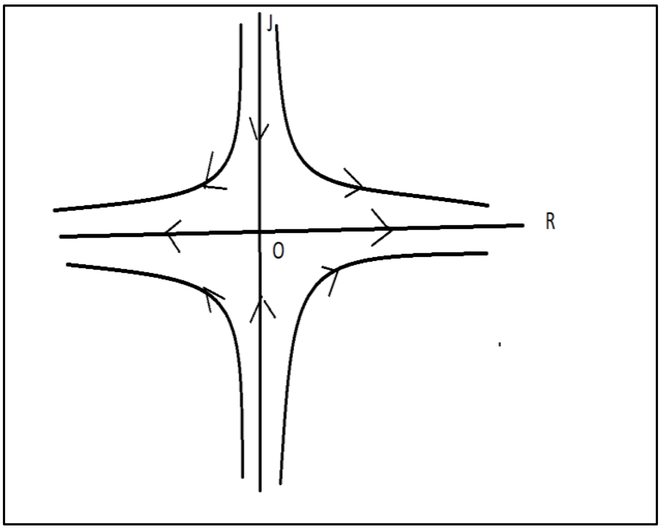
Example (1): Let $\lambda_1 = 1$ and $\lambda_2 = -1$ $i.e.$ $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $So\ R = c_1 e^t$ $and\ J = c_2 e^{-t}$.

If we plot it in R-J Graph then clearly we get two straight line solution corresponding to

$c_1 = 0, c_2 \neq 0 \Rightarrow R = 0$ $and\ J = c_2 e^{-t} \Rightarrow J \to 0$ $as\ t \to \infty$ $i.e.$ $J\ axis$. Similarly,

$c_1 \neq 0, c_2 = 0 \Rightarrow R = c_1 e^t \Rightarrow R \to \pm\infty$ $as\ t \to \infty$ $according\ as$ c_1 $is + or -$ $and\ J = 0$ $i.e.$ R $axis$. Direction is given as increasing t.

$If\ c_1 \neq 0$ $and\ c_2 \neq 0$ $then\ RJ = c_1 c_2$ which is a rectangular hyperbola. Using this we can draw the phase trajectory. Direction should match with straight line trajectory.



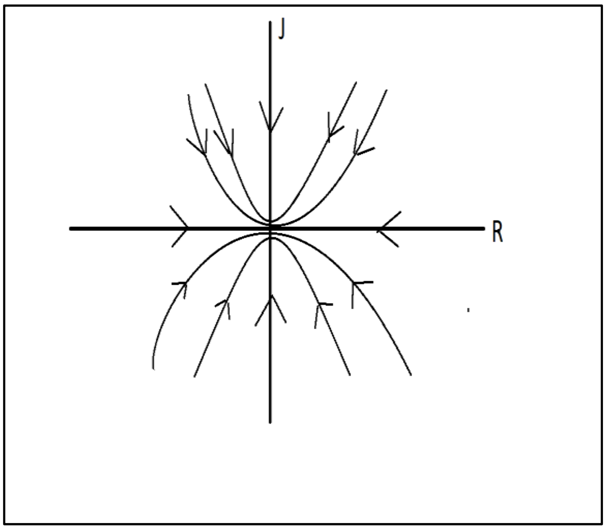


Fig 1

Fig 2

Example (2): Let $\lambda_1 = -1$ and $\lambda_2 = -2$ $i.e.$ $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$.

$So, R = c_1 e^{-t}$ $and\ J = c_2 e^{-2t}$. If we plot it in R-J Graph, then clearly we get two straight line solution corresponding to

$c_1 = 0, c_2 \neq 0 \Rightarrow R = 0$ $and\ J = c_2 e^{-2t} \Rightarrow J \to 0$ $as\ t \to \infty$ $i.e.$ $J\ axis$. Similarly,

$c_1 \neq 0, c_2 = 0 \Rightarrow R = c_1 e^{-t} \Rightarrow R \to 0$ $as\ t \to \infty$ $and\ J = 0$ $i.e.$ R $axis$.

$If\ c_1 \neq 0$ $and\ c_2 \neq 0$ $then\ R^2 = \frac{c_1^2}{c_2} J$ which is a parabola with axis as J axis (along greater eigenvalue) and tangent at Critical point as R axis (along numerically lesser eigenvalue). Using this we can draw the phase trajectory. Direction should match with straight line trajectory.

Similarly, for $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ phase diagram will be same as fig 2 but the direction will be opposite.

Let us take a matrix which is not the diagonal matrix.

$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$.

Here characteristics equation is $\lambda^2 - 4 = 0; i.e. \lambda = \pm 2$. Eigenpairs can be found as: $\left[2, \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right]$ and $\left[-2, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$.

So, the solution is $X = c_1 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Again, $c_1 = 0, c_2 \neq 0 \Rightarrow R = c_2 e^{-2t}$ and $J = -c_2 e^{-2t}$. Hence straight-line solution is given by $R + J = 0$ line, direction towards the origin.

Similarly, $c_1 \neq 0, c_2 = 0 \Rightarrow R = 3c_1 e^{2t}$ and $J = c_1 e^{2t} \Rightarrow R = 3J$ line. Direction away from origin (CP).

Other trajectories are accordingly. See fig 3:

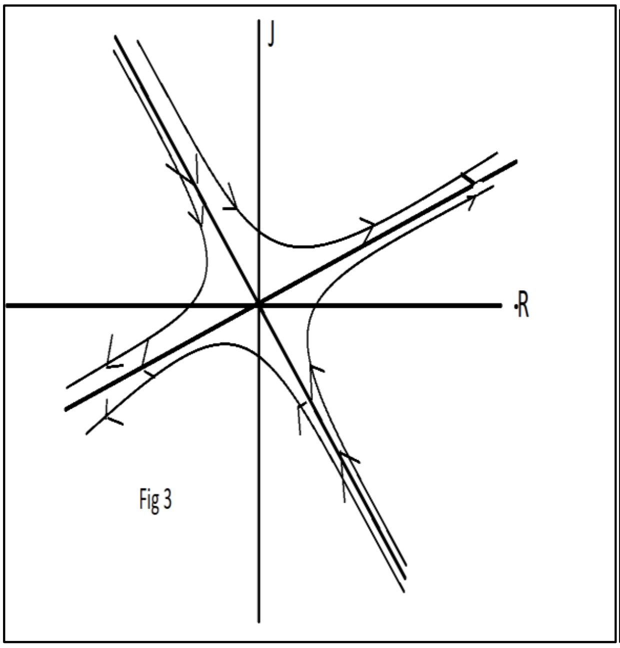


Fig 3

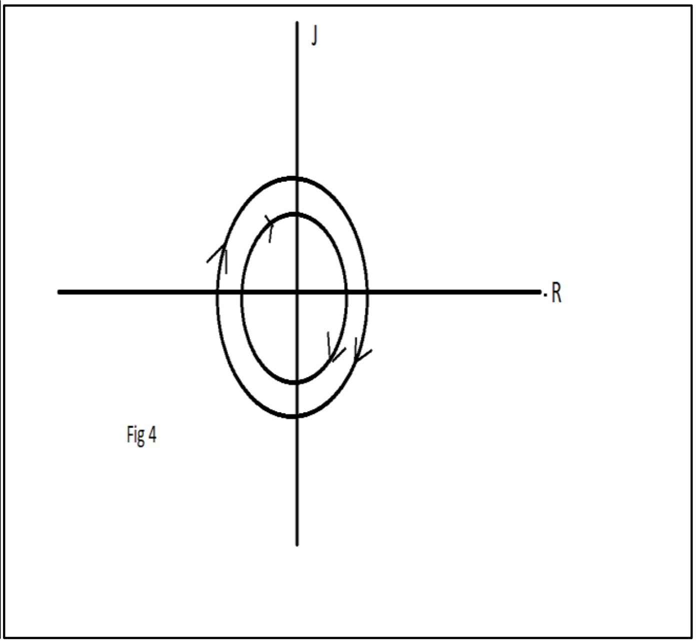


Fig 4

Now we can analyse these three figures:

If our starting point is in first quadrant i.e., R > 0 and J > 0 in fig 1 then J is decreasing and R increasing. Ultimately, as $t \to \infty, J \to 0$ & $R \to +\infty$, which means it is a one-sided love from Romeo, whereas in fig 2 both R and $J \to 0$ as $t \to \infty$ that means it is practically a no love no hate relationships or a cold relation. But in fig 3 if starting point is above R=3J line in the 1st quadrant then as $t \to \infty$, $both$ R & $J \to +\infty$ could be called a love fest.

It may be noted interestingly that depending on the starting point say in 2nd quadrant left of R+J=0 or right of the line fate changes absolutely may be a love fest or may be ultimate hate as $t \to \infty, both\ R\ \&\ J \to -\infty$.

So, if we study the matrix and their eigenpairs we can conclude the ultimate status of their love relationship depending on their initial condition.

Let us consider the case where eigenvalues are complex conjugate:

Let $A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$. Here eigenvalues are$\lambda = \pm i\beta$ and eigenvector corresponding to $i\beta$ is

$$V = \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix}\ then\ (A - i\beta I)V = 0 \quad gives - i\beta v_{11} + \beta v_{12} = 0 \Rightarrow v_{12} = iv_{11}\ ,$$

$Or\ V = \begin{pmatrix} 1 \\ i \end{pmatrix}$ can be taken as eigen vector. So, solution of $\dot{X} = AX\ is\ X = e^{i\beta t}\begin{pmatrix} 1 \\ i \end{pmatrix}\ or$

$$X = (cos\beta t + i\sin\beta t)\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \cos\beta t + i\sin\beta t \\ -\sin\beta t + i\cos\beta t \end{pmatrix} = \begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix} + i\begin{pmatrix} \sin\beta t \\ \cos\beta t \end{pmatrix}.$$

So, $\begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix}, \begin{pmatrix} \sin\beta t \\ \cos\beta t \end{pmatrix}$ are two independent solutions. Hence complete solution is

$$X = c_1\begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix} + c_2\begin{pmatrix} \sin\beta t \\ \cos\beta t \end{pmatrix}.$$

Here we will not get any straight-line solution but, $c_1 \neq 0\ , c_2 = 0 \implies R = c_1\cos\beta t\ \ and\ J = -c_1\sin\beta t \implies R^2 + J^2 = c_1^2$ and $c_1 = 0\ , c_2 \neq 0 \implies R = c_2\sin\beta t\ \ and\ J = c_2\cos\beta t \implies R^2 + J^2 = c_2^2$.

Both are circular solution.

Phase plane diagram will look like fig 4. To get the anticlockwise direction we choose cleverly $c_1 > 0\ ,\ \beta = 1\ , t = 0\ \Rightarrow a\ point\ (1,0) on\ R\ axis\ and\ t = \frac{\pi}{2} \Rightarrow a\ point\ (0,-1)$ on negative J axis. On increasing t it will from (1,0) to (0, -1) which is counter clockwise direction.

Another example:

$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$,here $\lambda = \alpha \pm i\beta$. Corresponding to $\lambda = \alpha + i\beta$ let $V = \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix}$ be the eigenvector. Then $(A - \lambda I)V = 0 \Rightarrow -i\beta v_{11} + \beta v_{21} = 0 \Rightarrow v_{21} = iv_{11} \Rightarrow V = \begin{pmatrix} 1 \\ i \end{pmatrix} .So$

$X = e^{(\alpha+i\beta)t}\begin{pmatrix}1\\i\end{pmatrix}$, or $X = e^{\alpha t}(\cos\beta t + i\sin\beta t)\begin{pmatrix}1\\i\end{pmatrix} = e^{\alpha t}\begin{pmatrix}\cos\beta t + i\sin\beta t\\ -\sin\beta t + i\cos\beta t\end{pmatrix}$

$= e^{\alpha t}\left\{\begin{pmatrix}\cos\beta t\\ -\sin\beta t\end{pmatrix} + i\begin{pmatrix}\sin\beta t\\ \cos\beta t\end{pmatrix}\right\}.$

So, $e^{\alpha t}\begin{pmatrix}\cos\beta t\\ -\sin\beta t\end{pmatrix}$ and $e^{\alpha t}\begin{pmatrix}\sin\beta t\\ \cos\beta t\end{pmatrix}$ are two independent solutions. Hence $X = c_1e^{\alpha t}\begin{pmatrix}\cos\beta t\\ -\sin\beta t\end{pmatrix} + c_2e^{\alpha t}\begin{pmatrix}\sin\beta t\\ \cos\beta t\end{pmatrix}$.

To draw phase plane diagram let us take $c_1 \neq 0$, $c_2 = 0 \Rightarrow R = c_1e^{\alpha t}\cos\beta t$ and $J = -c_1e^{\alpha t}\sin\beta t$.

$\Rightarrow R^2 + J^2 = c_1^2e^{2\alpha t}$ which can be considered as a circular spiral with increasing radius $c_1e^{\alpha t} \to \infty$ when $\alpha > 0$ and for $\alpha < 0, c_1e^{\alpha t} \to 0$ as $t \to \infty$ i.e., decreasing radius.

Here four possible phase diagrams are there.

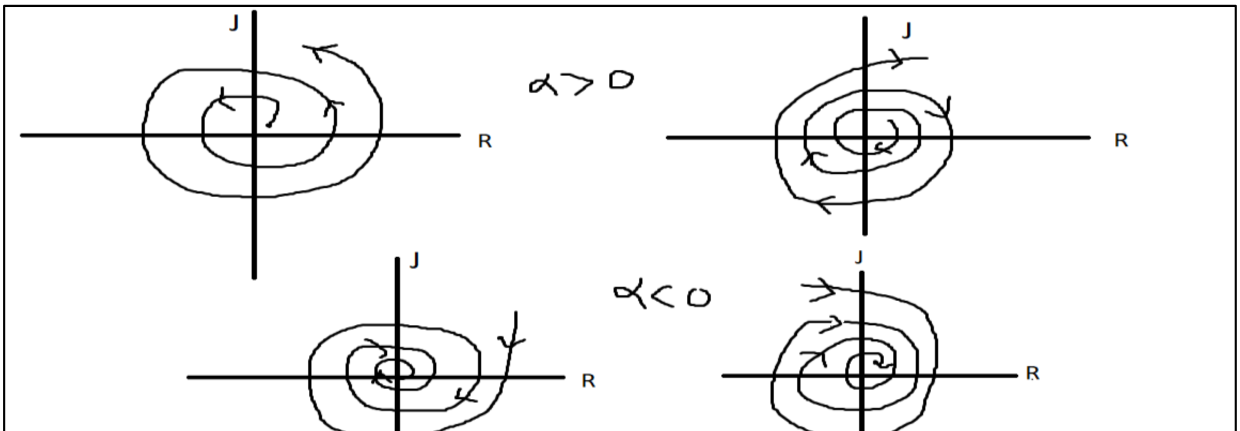


Fig 5

If we analyse Fig 4 then clearly love of Romeo and Juliet will sinusoidal i.e., increase and decrease alternatively but never extinct or increased beyond boundaries (Natural ??) whereas in Fig 5 it will increase and decrease alternatively but increases beyond boundary if $\alpha > 0$ but ultimately dies out if $\alpha < 0$ (Reality??).

Finally, we take $A = \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$, this is the Jordan Canonical form where both the eigenvalues are λ and only one eigenvector we have, which is $\begin{pmatrix}1\\0\end{pmatrix}$ here. Here we solve the equation directly instead of using generalised eigenvector.

Equations are $\dot{R} = \lambda R + J$ and $\dot{J} = \lambda J$ which gives $J = c_2e^{\lambda t}$. Putting in the equation we get,

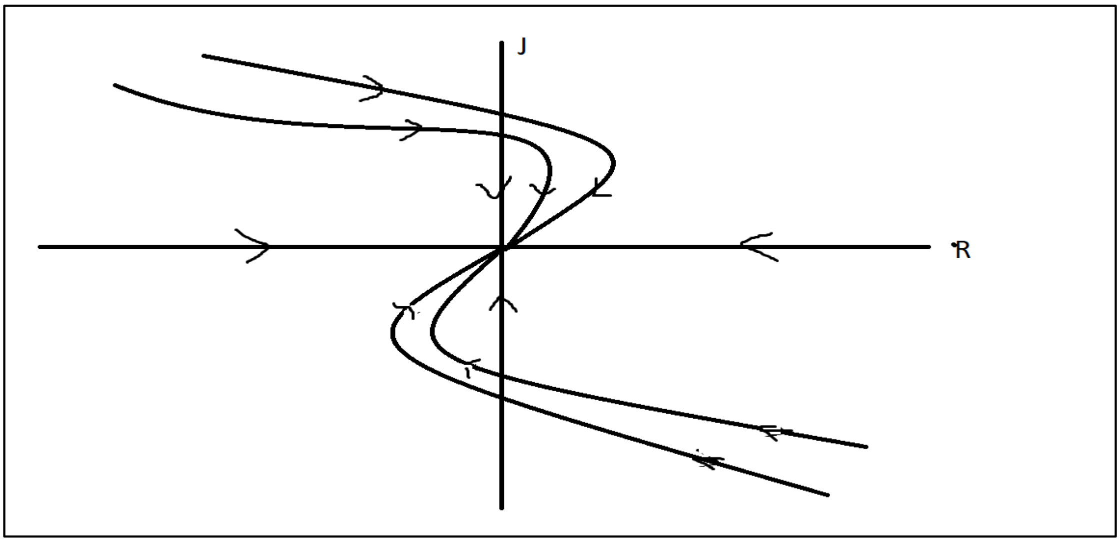
$\dot{R} = \lambda R + c_2 e^{\lambda t}$ which is a linear equation with integrating factor= $e^{-\lambda t}$. Hence solution becomes $Re^{-\lambda t} = c_2 t + c_1 \Rightarrow R = e^{\lambda t}(c_1 + c_2 t)$ & $J = c_2 e^{\lambda t}$.

So, $X = c_1 e^{\lambda t}\begin{pmatrix}1\\0\end{pmatrix} + c_2 e^{\lambda t}\begin{pmatrix}t\\1\end{pmatrix}$. In fact, $\begin{pmatrix}1\\0\end{pmatrix}$ *and* $\begin{pmatrix}t\\1\end{pmatrix}$ are generalised eigenvectors of A.

Let us take $\lambda = -1$. Hence solution is given by $X = c_1 e^{-t}\begin{pmatrix}1\\0\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}t\\1\end{pmatrix}$.

Then $\frac{dJ}{dR} = \frac{J}{R-J}$.

Phase portrait is given by:



Observe that trajectories have a tangent perpendicular to R axis at a point on the line R=J. For $\lambda > 0$ direction of trajectory will be opposite.

Hence to predict the fate of romantic love relationship, it depends on the matrix and its eigenpair along with initial values of R and J. In the long run, love for each other may increase without boundary or extinct. It may be that one be silent(constant) but other increases or extinct. If both the eigenvalues are negative and having negative real part, then certainly love will be extinct from both side in the long run which matches with our common experience. But if eigenvalues are positive then it could be love fest or absolutely hate from both side or absolute love from one side and absolute hate from the other side depending on their initial status. If eigenvalues are of opposite sign, then either it is an ultimate romance or absolute hate from both side. We know that most common dialogue between two lovers is like this:

<u>Romeo</u>: I love you and <u>Juliet</u>: I love you too.

Or

<u>Romeo</u>: I love you; do you love me? and <u>Juliet</u>: Yes, for ever I love you.

From the above discussion, conclusion is that they should ask each other that: Can we make our love eigenvalue complex with positive real part ??(!!!). Because that is the only case where in the long run it will be a love fest, though some love-hate relationships are there in the short run whatever be the initial status.

The Daddy of Big Numbers(Rayo's Number):Rayo's number is a large number named after Mexican associate professor Agustín Rayo ,which has been claimed to be the largest (named) number.It was originally defined in a "big number duel" at MIT on 26 January 2007.The definition of Rayo's number is a variation on the definition:The smallest number bigger than any finite number named by an expression in the language of first order set-theory with less than a googol (10^100) symbols.

PUZZLES, GAMES AND MATHEMATICS

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With the school day memories still lingering in the mind when a young student first encounters a strange object like Groups, he or she is sure to get disappointed or at least taken aback, thinking is it the same Mathematics I have studied? Or is it the one that I had yearned for or may be is it Mathematics at all? The more you try to dig into relating the school maths with college maths the more you get frustrated. However, its seen, the more you ignore and swallow like a quinine pill the strange or monstrous objects like groups, homomorphism, the better you understand and succeed. But does this answer the question, is school maths different from college maths? No, not at all. You bisected an angle in 8th standard or so. What about trisection. May be in college. But instead of trisecting angle you learn more about trisecting your own brain in Algebra, Analysis, and Metric spaces. Natural to ask why? I mean why Algebra and why not angle? The reason is you keep trying and keep on trying and ultimately after nearly wasting few days and plenty of pages you are at a loss. The reason is actually no one can trisect an angle with a non-calibrated ruler and compass.

The first reason that strikes your mind is may be till now no one has been able to do it. But what is the guarantee that no one will be able to do. This idea is difficult to conceive. Am I God that I can say what will happen in future? But it is very simple notion you get confused because I am putting it in this way. Now I say this statement. Suppose you appear for the group action exam which is 100 marks paper. I claim no one in this world can get more than 100 in this paper. Still, you see you can ask the same question ok till now no one has got but what in near future? But this doesn't strike your mind since you know more than 100 is not possible. Now let us take a set.

$S = \{2^{3^n} + 1 | n \in \mathbb{N}\}$. However, you try you cannot find a prime number in S. This also does strike your mind, since you know each term is divisible by 3 and greater than 3. But in case of trisection where is such impossibility coming? Because you are not rephrasing it mathematically. We try to make such attempt.

Let $S = \{\theta | \theta \text{ can be trisected}\}$. So may be 60^0 is not a member. So, one question can be to show this fact. So firstly, you must translate what do I mean by trisecting by ruler and compass. This means we must define constructability first. So let us start with few points in the beginning. Then joining as many as we want and drawing circles with radii equal to their distance of separation the new points are created by their intersections. Now given say 10 points you can easily use trigonometry and plane geometry to find all possible intersection of these arcs and straight lines and you get co-ordinate for the new points. Now replace 10 by n you get some more. So, you try to identify that which new element can be produced from the old points and what is the relation. Then you gradually understand some condition must be satisfied for an element to belong to S. And then 60^0 cannot be trisected is rephrased mathematically. But how to show whether 60^0 is a member or not. Then if you keep trying elementary methods won't work. This will force you to solve equations and dealing roots. And then you see group theory enters. And you need to learn a lot of theory (group, rings fields, Galois) if you want to understand the proof. So, you were asking at kinder gardens why we are learning Alphabets....

Difference between Mathematics and reality: Suppose a magician comes and claims forget your proof I can trisect any angle. I don't even need a compass. You can't ignore. Must accept his challenge. Now you give him 60^0. He takes the ruler and draws a line, and you check it is trisected. This is possible mathematically also. There is a point which if you join the angle becomes 20^0.

Its difficult but not impossible. Now every time you give an angle, he can choose a line that trisects the angle. So, what goes wrong? This is where Mathematics and reality gets separated. Mathematically, constructible means you must provide an algorithm to trisect. More precisely it means if you choose any point on the paper you must be able to explain which point is it. Magician must tell the one I have in my mind. But that is not acceptable. I explain why. I tell the Magician maybe you can construct the angles I give to you but how do you guarantee that you can do for all angles. Magician argues well then you give me one. I tell I can give you only finitely many that can never cover all cases. Then Magician tells then you can't say I cannot, you can at most say not all cases are tested. But this is not what is called constructible. Constructible means the point must be formed through intersecting lines and arcs placing compass needle on already constructed points or joining those points. You cannot choose to draw a circle at any arbitrary point in the paper or join it to another point. But what if the point is arbitrary? Well, up to my knowledge this is not disproved in Mathematics. But it can be rephrased mathematically though.

Another exciting thing may be is $\sqrt{2}^{\sqrt{2}}$ a rational number? It seems it is irrational. But can one prove it? You might think first it's easy. But then as you try to investigate you see you are stuck at the first step. But this obvious question which even can be understood at 9^{th} or 10^{th} standard and is very natural to ask turns to be a problem again in Galois Theory where we first go from transcendental and then to irrational. But imagine at the beginning will this problem be motivation to learn groups? A mere mention will only force the students to accept it instead of visualization.

Now comes our age-old games and we see how these works mathematically. Its Bijoya Dosomi, the women are playing with sindoor. So, suppose near a pandel 70 women are playing sindoor. It's a custom that if someone puts vermillion you are also supposed to reciprocate it. Then there are at least two women who have played vermillion with the same number of women. Like may be the first women has played with three other women. The second women may be with 5 and so on. This can be formulated in this way, if we consider the women as dots and playing sindoor indicates joining a line between the dots.

Then the question rephrases to, there are two dots with same number of lines passing through them. This has a name in Maths. It is said as given a graph with 40 vertices there exist two vertices of the same degree.

Next comes a game. A and B plays a game. A lay downs lottery tickets marked from 1 to 100. B cannot see the number. B pulls two tickets and A confirms whether they are consecutive or not. B must pick two consecutive tickets and is given 98 chances. Can be B ensure a win?

You can try and I won't be a spoiler. I just rephrase it in another way. This problem rephrases to give a graph of 100 vertices and 98 edges you can number the vertices with 1 to 100 so that no two consecutive vertices share an edge. [This is if B can't ensure a win]

Now let us move to some more popular problems.

1. A student takes tuition from a teacher and has to pay him fees. Now the teacher is a wicked one and tells the student your fees will be decided on the number of classes you were mentally absent or was asked to unjoin. However, the fee is between Rs1 to Rs 1000. So, the student doesn't know the actual fees. He asks his mother for 1000 bucks, but his mother is suspicious. So, she gives her son 10 sealed envelopes and writes the amount on the top of each envelop. The teacher will take as many sealed envelopes as he wants. So, what must be the amount the mother puts in each envelop so that the teacher gets his exact payment?

2. A magician asks you to select an item from a list of 60 food items. After you select, he gives you six cards and asks you to return the cards where you find your selected item. You give him and he tells you the item you chose. Will the magic still work for 5 cards?

3. A iron chuck weighing 100 Kilos dropped from crane near a construction area and was broken into five pieces. Can you tell what were the broken parts so that you ask for any weight between 1 to 100, these weights can be used for a weighing machine (common balance).

Solution:

The first problem. Not only 1000 we can go till Rs 1023. His mother puts 1 rupee in first box, 2 in second, 4 in third and so on. Why this works? We use induction. Our claim is that for n packets filled in this manner any amount from 1 to $2^n - 1$ can be paid. Clearly, true for $n = 1$. Now let it be true for $n = k$. So for $k + 1$ packets the range is upto $2^{k+1} - 1$. Let $y \in 2^{k+1} - 1$. If $y < 2^k$ we are done with first k packets by induction. If $y \geq 2^k$, then take the $k + 1^{th}$ packet. Then we need to fill for $x = y - 2^k$. But $y < 2^{k+1} \Rightarrow x < 2^k$. So x is achieved through induction. Hence, we add those packets which make up for x.

Now this cannot be achieved with 9 packets. Why? May be the same denominations won't work but why not in a different method. I pose the problem as there exists no method to make 9 packets to give all possible values from 1 to 1000. Again, we argue as follows: The only thing the teacher can do is choosing the envelops. Once the envelops are chosen the total amounts gets fixed since you can't break the seal.

How many possible choices. This is school level combinatorics. $2^9 = 512$. Now it might happen that different combination of envelops can have same value in total but same set of envelops have a fixed amount. So, at max you can have 512 different values. How can you have for 1000?

The second one is a replica of the first. As for the third because it's a balance you can put weight on both the sides. Now each weight has three options either its put on the same side of the balance or on the other side or not put on the balance. Rest you can do. And show the same cannot be achieved with four pieces.

Now let's see a game using linear algebra. A king has kept his treasures in a secret place and has encoded that with 7 numbers. Now he calls all his sons and gives a hint in such a way that any seven brothers can decipher the code but no six can. The king has 50 sons. How he codes and what is the hint he gives?

Solution:

Where does Linear algebra comes in right? Well, apparently it has no connection. But think what hint the king can give. No six can decipher yet seven can, implies that from any seven you can find the seven numbers yet six can't give the numbers. So, some sort of uniqueness in solution is apparent here. Now your mind recollects linear equations. In linear Algebra interpretation isn't it like seven is the rank of some matrix! Well, think about it. A last hint Vandermonde's Determinant.

In a casino, at Table number 32, 4 players are asked to take their seats so as to form a circle. They eyes are blindfolded. Now in front of each one of them one card is kept. The cards are colored green on one side and black on the other side. In each round the players will be asked if they want to skip the round or flip the card. Players answer "SKIP" or "FLIP". And the cards are flipped accordingly. After each round if all the cards either show green or if all shows black the players win. If not, the table is rotated arbitrarily. And round 2 begins. The host wins if 16 rounds are completed, and the players haven't won so far. What strategy the players will use?

There are many ways to view this problem. Firstly, you can consider SKIP as 0 and FLIP as 1. So, each round is a four tuple with entries 0 or 1. So question is starting from any tuple can we reach (0,0,0,0) or (1,1,1,1) through flips. Problem is the players are blindfolded. They do not know from where they are starting. So even if they keep flipping how to guarantee to reach (0,0,0,0) or (1,1,1,1). This is achieved if they apply all possible combinations of SKIP and flip i.e., 16 times. So now the problem is can that be achieved in 16 moves. Seems easy but wait! When the table is turned again you are back to the pavilion. Since the combination you have in mind may have already been achieved due to this rotation. For example, may be your choice is FSFF. But the table was rotated. If it was rotated to places, then the actual permutation might be FFFS which might have been considered before. So now you can understand this is like cyclic permutation where (1234) is same as (2341), (3412), (4123). So now might be you are sensing something as orbits. For example, FFFS is same as FFSF, FSFF and SFFF since the table might turn up to any position so these turns are falling in the same class. We explain this more clearly in the following example.

In a round table conference 20 glasses of juices of 4 distinct types (orange, lemon, mango, grape) are served to the delegates. How many ways can you arrange the drinks?

This is a combinatorics problem. We must pour the juices in the glasses irrespective of any order. We will easily get the total number of possibilities. But we must understand, that being a round table rotation will experience the same pattern. So cyclic permutation. But problem is juices have same colors. If you try to do it through elementary tricks, you end up only with over or under counting. So how to tackle this? So, we can think each combination as a 20-tuple with entries as 0,1,2,3 (four types) and we say two tuples same if after rotating few places they remain the same tuple. Like (00000111112222233333) is same as (22222333330000011111). Hence, we can think it as an action of $\mathbb{Z}_{20}$ on $\mathbb{Z}_4^{20}$ given by $\big(i,(a_1,a_2,\cdots,a_{20})\big) \mapsto (a_{1+i}, a_{i+2,}\, a_{i+3,} \ldots \ldots.)$ with addition is taken in modulo 4. So, our answer is to find number of distinct orbits given by Burnsides Lemma.

Some logical innuendo.

A teacher announces I will take a surprise test this week. Show the teacher cannot take a surprise test.

If the teacher doesn't take the test on Friday, then the students know that Saturday is the exam day. So, it's no longer surprise. So, the teacher cannot take the test on Saturday. If the teacher doesn't take the test on Thursday the students know since the teacher cannot take it on Saturday, so the teacher has to take it on Friday. Then the student cannot take surprise test on Friday as students get to know. Proceeding inductively, we are done.

So far so good. But what if the teacher takes a test on some day? Is there a Mathematical inconsistency? No. It is mathematically correct. Then why this discrepancy?

Let's do another problem. In a conference 40 participants had given lecture demonstration. The judges discussed and told that at least one of the participants performances is unsatisfactory. From the next day whoever gets to know is disqualified leaves the conference. However, each participant knows who among the others are disqualified but doesn't know his own fate. But it is found after some day few participants left. How did they come to know?

Apparently, this is impossible. But see logically if you find no other participant is disqualified then it is you since at least one participant is removed. But if you find one person among the 39? Then you might be or might not be. But see the other can also see you. And if you are not ill-fated, the next day he won't come. So, if the next day he doesn't arrive then you are sure he was the only one. But if he shows up next day it means you are also disqualified. Hence, we proceed. But don't things seem odd in your mind? What if the person is not intelligent? Even if you assume him intelligent does that ensures he will think exactly in this fashion?

So is the surprise test problem. You are assuming the teacher is thinking exactly like the students. This is where again mathematical rephrasing faces a serious problem.

Lastly, we come up with some exciting problems like the 15 puzzle and the Rubik's cube.

15 puzzle: We are given a grid like this.

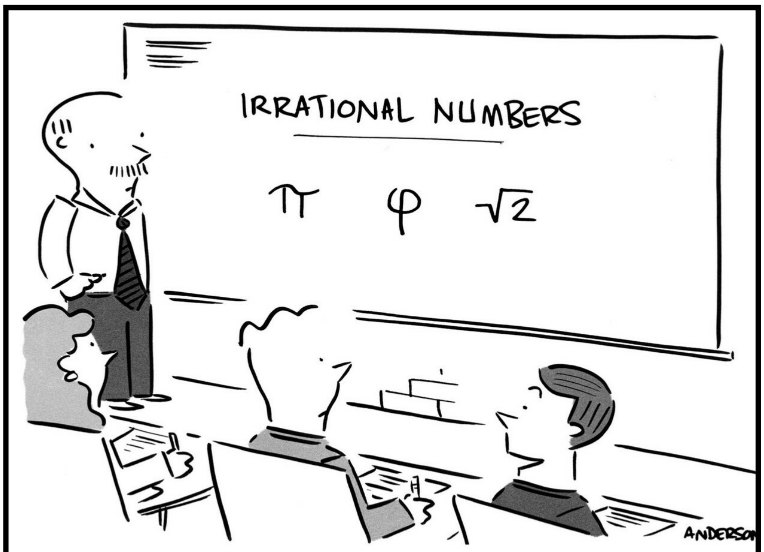
10	2	5	14
1	12	4	8
9	3	15	6
11	13	7	

By moving the pieces in the empty gap can arrange the blocks so as to get a grid like.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

How to solve this? But before even solving the question which comes in mind is, is it at all possible? Can we rephrase it mathematically? If we carefully observe our question is whether from one permutation, we can move to another by a sequence of moves? But how can we translate moves? If we can the blank space as 16, then basically moving a piece in blank place is involving interchange of the two pieces. So basically, a transposition. So now the question leads to whether any permutation can be reached through transpositions. We also observe number of movements must be even as the blank place doesn't remain at the last place at the end of the game. So, it must move as many steps up that many steps down and as many steps left so many steps right. So basically, every permutation must be an even permutation, mathematically must belong to A_{16}. Rubick's cube clearly can be thought as a 3D manifesto is also understood through group actions and we name the group as Rubick's cube group. This group has a huge order and very difficult to investigate though an explicit formula exists.

We end by posing another puzzle. Two friends A and B are asked to think of a positive integer each. Now they tell the number to C. C writes two numbers on the board. One is the sum other is arbitrary. He then starts with A and asks if A can tell which one is the sum. If A passes, then B gets the turn. The game continues until one of them tells the answer. How is it possible for either A or B. They can only utter YES or NO.



"So no matter what, you can't make them fractions?! Those are numbers I can get behind!"

Articles

NEWTON-PEPYS PROBLEM : FROM A DIFFERENT VIEWPOINT

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"*Laws of probability, so true in general, so fallacious in particular*" – Edward Gibbon

Probability theory is nothing but study of uncertainty using mathematics and logics. When intuition meets theoretical calculations, *magic of probability emerges.*

Let us consider a *Die game*- the die has a certain property. '*Six*' is more likely to occur when we throw the die (may be a weighted die) i.e., the die is biased towards '*Six*'. Now, a participant has two options to play and win the prize. The options are –

1. Roll the die six times independently, if at least one '*Six*' occurs, win.

   Or,

2. Roll the die twelve times independently, if at least two '*Six*'s occur, win.

As, it is known that '*Six*' is more likely to occur, any lazy person will choose option-1 to play the game. But, is it really worthy to choose option-1?

It's very similar to one of the famous historical problems, called 'Newton-Pepys Problem' (1693). Samuel Pepys, former president of Royal Society of London, wrote a letter to Sir Isaac Newton regarding a *gambling problem.* The problem is given below –

"*Which of the following three propositions has the greatest chance of succes*

A. *Six fair dice are tossed independently and at least one "6" appears.*

B. *Twelve fair dice are tossed independently and at least two "6"s appear.*

C. *Eighteen fair dice are tossed independently and at least three "6"s appea*

Samuel Pepys thought that C is more probable. But Newton convinced with his calculations and logics that A is more probable. On the first go, may also think that all are equiprobable or C is more probable (Since, m trials, more chances to win). But actually, A is more probable.

Mathematical Justification Using Random Variables :

Newton uses first principle to calculate these probabilities as then concep probability was only at its infancy. We will use random variables to discuss problem.

Let, X : Number of Sixes in N independent throws of a die (not necessarily fa Clearly, *X ~ Binomial(N, p).*

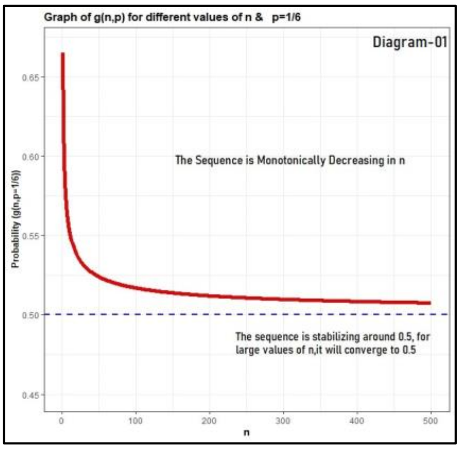
Where, p = Pr('Six' appeared in a single throw of the die). For fair die, p =

Let us define, g(n, p) = Pr(X $\geq n \mid N = 6n, p$) = $\sum_{x=n}^{6n} \binom{6n}{x} p^x (1-p)^{6n-}$

n $\epsilon\ \mathbb{Z}^+$

Note that, g(1, $\frac{1}{6}$), g(2, $\frac{1}{6}$), g(3, $\frac{1}{6}$) are the probabilities of Pepys' interest!

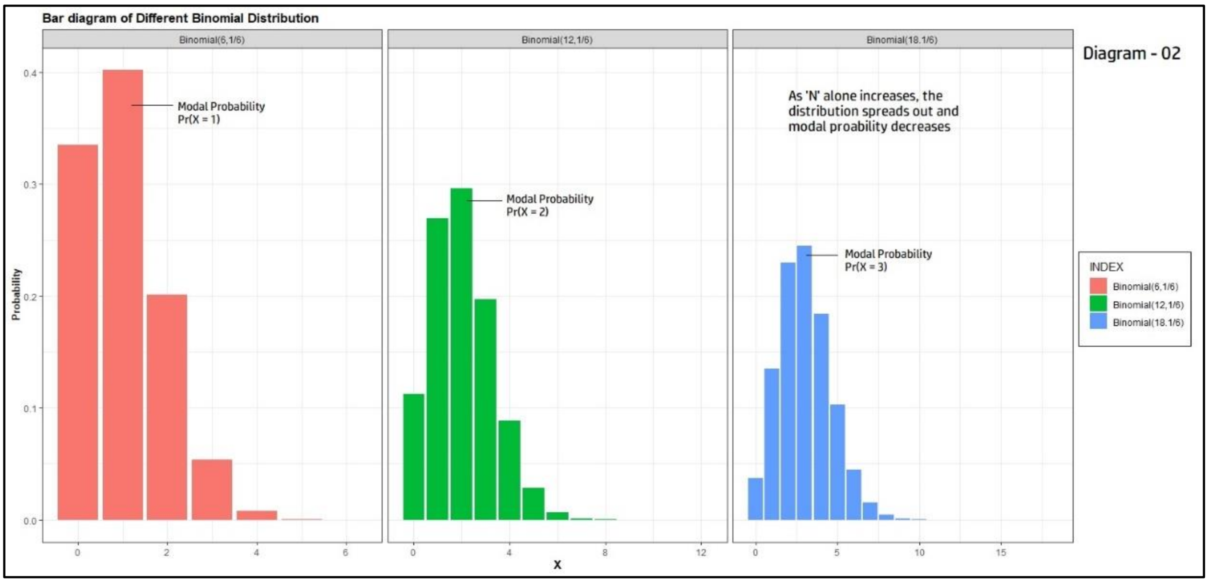
If we draw the graph of g(n, p = $\frac{1}{6}$) for different values of n, we will see tha n increases the sequence monotonically decreases but is bounded below by



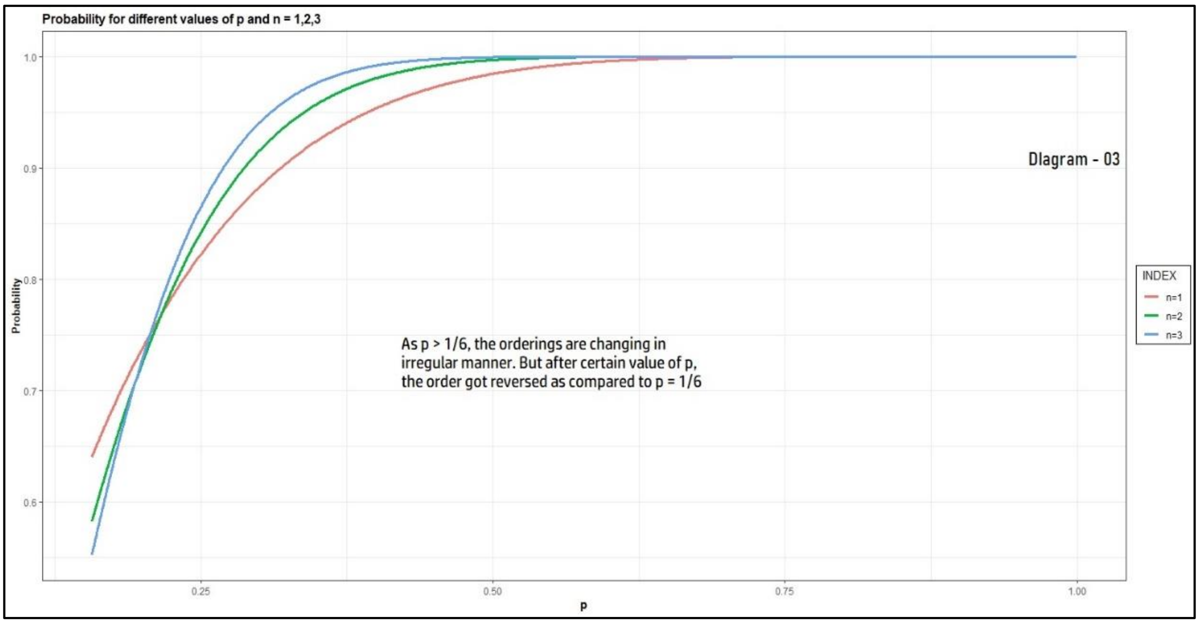
We know that if X ~ Binomial(N, p) , Np is the mean of the distribution. Also, if (N+1)p is not an integer, then [(N+1)p] is mode of the distribution. In our problem, N = 6,12,18 and $p = \frac{1}{6}$. So, 1,2,3 are the respective mean and mode of the distribution. Also, they are median. So, mean, median, mode coincide, though the distributions are positively skewed. It can be shown that for N = 6n & $p = \frac{1}{6}$.

$$g(n, p) \approx \frac{1}{2} + (0.4)\ Pr(X = n \mid N, p)$$

For binomial distribution, as N alone increases the distribution spreads out. As a result, in our case the modal probability Pr(X = n | N, p) decreases => g(n, p) decreases. It is obvious from the graph below.



But, it's not always the case! If $p > \frac{1}{6}$, the ordering may change.



It is evident from the graph that after certain value of p ($> \frac{1}{6}$), we can see different types of ordering. But again, after certain value of p (approx. 0.22), we will see a monotone ordering of g(1,p), g(2,p) & g(3,p), which is completely opposite of $p = \frac{1}{6}$.

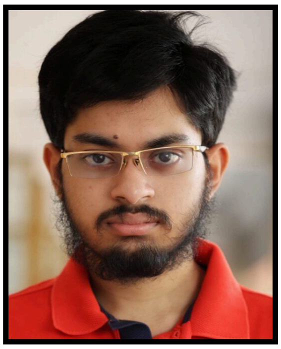
Now, we can visualize what happens to the *Die game* proposed at the beginning. For the die $p > \frac{1}{6}$. Following the same logic, it is now clear that the decision of the lazy person is not always worthy in terms of probability.

It is to be noted that, the calculations of Newton were *correct* but the logic was *wrong*, which was explained by Prof. Stigler in his article '*Isaac Newton as a Probabilist*'.

A further generalization of the problem is given by Chaundy and Bullard (1960).

Imagine you're holding a postcard in your hand, on one side of which is written, "The statement on the other side of this card is true." We'll call that Statement A. Turn the card over, and the opposite side reads, "The statement on the other side of this card is false" (Statement B). Trying to assign any truth to either Statement A or B, however, leads to a paradox: if A is true then B must be as well, but for B to be true, A has to be false. Oppositely, if A is false then B must be false too, which must ultimately make A true....

HOW TO KILL THE KING ?



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Background:

Takeda Kenshin is a renowned samurai of the Minamoto clan. He has earned his reputation by winning many battles single-handedly and assassinating evil warlords, kings, and samurais all over Japan. He is the "G.O.A.T" in his clan and his clan members are willing to lay out their lives for him without a moment's notice.

In the present day, he is travelling with 56 of his most trusted clan members on a mission to assassinate Toyotomi Nobunaga, one of the most hated Kings of Japan.

However, one of his clan members has betrayed him and has already ratted out Takeda's route and plan of action to the king in return for 6,000 gold coins.

Consequently, on the 6th day of his journey towards the King's palace, Takeda's team is ambushed and is captured by the King's guards.

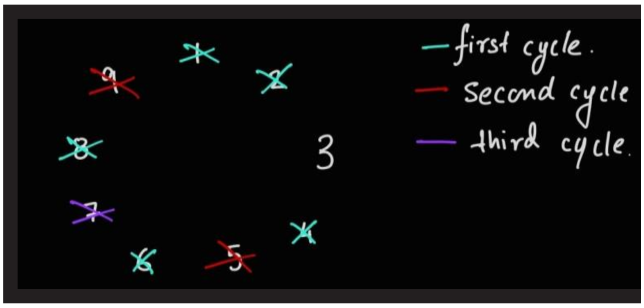
The King, known for his eccentric ideas, has planned something very evil for Takeda and his team. He calls this a game and has named it 'The Killer's Circle.'

The Killer's Circle:

Takeda and his team are brought to a huge hall with a huge round table with 57 similar iron chairs which have been numbered from 1 to 57. The king announces the rules of the game:

- Takeda and his team have to sit around the table with their swords in their left arm.

• The game would start from the samurai on chair 1.



• At their turns, every samurai must kill the samurai immediately to his left.

• This cycle would continue until only one samurai is left sitting at the table.

• This samurai would have the choice of killing himself or challenging the king to a duel which, if he wins, will guarantee his freedom.

<u>Takeda's Challenge:</u>

The King has given Takeda's team 2 minutes to decide the order in which they sit. Without a moment's hesitation, his team has agreed that Takeda must win, finish the King and avenge his brothers. With time ticking away swiftly, where should Takeda sit to earn the duel with the King?

<u>An example of the rules:</u>

Let's consider a case of 9 players. As can be seen from the diagram, 1 kills 2, 3 kills 4, 5 kills 6, 7 kills 8, 9 kills 1, 3 kills 5, 7 kills 9, 3 kills 7 and 3 is the winner.

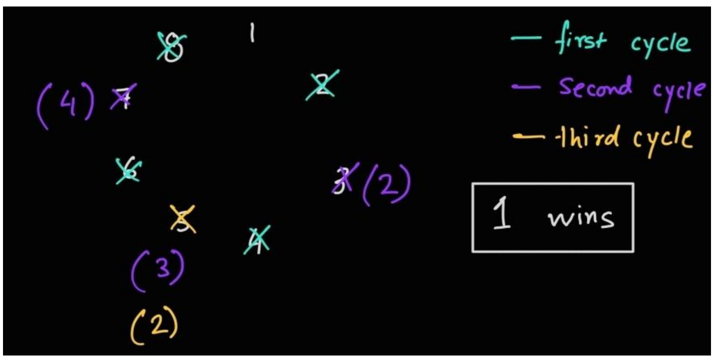
<u>The Solution:</u>

Any natural number (N) can be expressed as the sum of a power (k) of 2 and some other number (p).

$$\text{So,} \quad N = 2^k + p$$

How many cycles would be needed to complete the game?

Consider the case when $N = 2^k, p = 0$.



After every cycle, half the number of players die as every odd-numbered player kills the even-numbered player with respect to 1 (considering 1 as the first chair). For example, 1 kills 2, 3 kills 4, and so on.

After the 1st, 2nd, 3rd, 4th, ..., 'k'th cycle, we have

$\frac{N}{2}, \frac{N}{2^2}, \frac{N}{2^3}, \frac{N}{2^4}, \ldots, \frac{N}{2^k} = \frac{N}{2}, \frac{N}{4}, \frac{N}{8}, \frac{N}{16}, \ldots, \frac{N}{N}$, players left. So, after k cycles, th game is up as we have only 1 player left.

Now, let $p \neq 0$, i.e, $N = 2^k + p$.

This is just an extension of the first case as after 'p' killings, the sam situation is repeated.

The 'p' killings are completed somewhere in the first cycle (as $2^k > p$). Now we have 2^k more players. Like the first case, we need 'k' cycles now to finis the game.

However, we aren't starting a new cycle after 'p' killings, we're just continuin the 1st cycle. So, the total number of cycles is still k.

Hence, in any case, we need 'k' cycles to complete the game.

The primary case: Let the number of players be a power of 2.

Thus, $N = 2^k$ and $p = 0$.

After the first cycle, every even player gets killed, i.e., 1 kills 2, 3 kills 4, ..., N 1(odd) kills N(even).

Now, we're left with $N_1 = \frac{N}{2} = 2^{k-1} = 2^{k_1}$.

The second cycle starts with 1 again. It is similar to the first cycle. Th number of players is a power of 2 and player 1 starts the cycle. Hence, ever player who is sitting in an even position now with respect to 1 gets killed. I player 1 is 1, 3 is 2, 5 is 3, 7 is 4 and so on and so forth. So, 1 kills 3 (new sea 2), 5 kills 7 (new seat 4), ..., (N-2) (new seat $\frac{N}{2} - 1$) kills (N-1) (new seat $\frac{N}{2}$).

Now, we're left with $N_2 = \frac{N}{4} = 2^{k-2} = 2^{k_2}$ players with 1 starting the nex cycle.

So, during the (k-1) cycles we have $\frac{N}{2}, \frac{N}{4}, \frac{N}{8}, \ldots, 2$ players left (which are a powers of 2) with 1 starting the next cycle every time as every cycle is simila due to N being a power of 2.

In the 'k' th cycle, we have 2 players with player 1 starting the game. Hence player 1 wins the game.

So, if $N = 2^k$, 1 shall win the game every time.

The general case of when $N = 2^k + p$, $p \neq 0$:

This case is just an extension of the first case.

Here, after 'p' samurais are killed, we will have the same situation as the firs case with N being a power of 2.

So, the first killer after ‘p’ killings will always win the game.
Now, how can we determine what position comes after ‘p’ killings?
Note that 1 kill requires 2 people (a killer and a victim). So, ‘p’ killings mean ‘2p’ people have already participated in the game (i.e., either have killed or have been killed). The seat immediately after 2p seats corresponds to that first killer for which the total number of players is a power of 2. Consequently, this is the $(2p + 1)th$ seat.
As we have seen in the first case, this player always wins.
Hence, in this game, the winning seat will be the $(2p + 1)th$ seat.

A limitation:

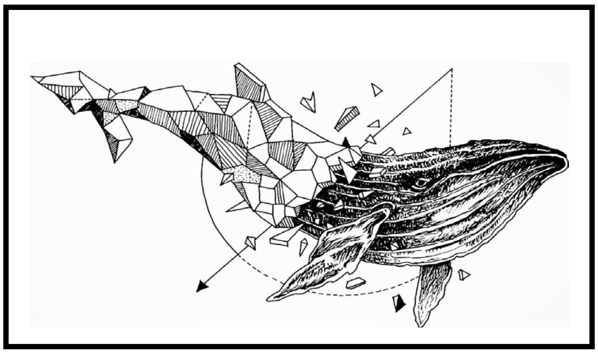
The formula doesn’t hold if p is greater than 2^k, i.e., 2^k must be the greatest power of 2 in N. This can be proved as follows:

$$\text{If } p > 2^k,$$
$$\text{then, } 2p > 2.2^k$$
$$\therefore 2p + 2p > 2.2^k + 2p = 2(2^k + p) = 2N \qquad \because 2^k + p = N$$
$$\Rightarrow 4p > 2N \Rightarrow 2p > N$$
$$\Rightarrow (2p + 1)(\text{winning position}) > N + 1$$

Which is not possible. So, $\mathbf{0 \leq p < 2^k}$

Conclusion:

Takeda quickly calculated $57 = 2^5 + 25$. Then he deduced that the winning seat is the $(2 \times 25 + 1) = 51$st seat. Taking his seat, Takeda wins the game with tearful eyes.
Toyotomi had no other choice than to challenge the vengeful samurai. After a quick one-sided fight, Takeda beheads the king with his katana and ends his malice once and for all.



BUILDING A MATHEMATICAL QUILT

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Mathematical quilting is a form of recreational mathematics that involves sewing mathematical patterns on a quilt. It can be a purely artistic venture or an effort to investigate a mathematical structure or phenomena in a creative way. Several professional and amateur mathematicians partake in it, sharing their creations on their blogs and other platforms.

One such quilt-like pattern arose out of the examination of relatively prime polynomials in x with coefficients in $\mathbb{Z}_2$. Before we generate the quilt ourselves, let's briefly look at $\mathbb{Z}_2$, define what it means for polynomials to be relatively prime, and understand how the Euclidean Algorithm can be used in our favor.

I. Introduction

The quotient ring $\mathbb{Z}_2$ consists of only two elements, denoted by {0,1}. It is defined with its usual addition as addition modulo 2 and multiplication as multiplication modulo 2. As a result of this, it has an interesting quality, i.e., addition is no different from subtraction.

Here, 1+1=0.

Thus, for any $n \in \mathbb{Z}, n + n = n(1 + 1) = n.0 = 0$.

From here, we should note that,

$$(n + m)^2 = n^2 + 2nm + m^2$$

$$= n^2 + (nm + nm) + m^2$$

$$= n^2 + m^2.$$

As a commutative ring with a multiplicative unity, where the non-zero element has a multiplicative inverse, we know that $\mathbb{Z}_2$ is a field.

Now, let's consider polynomials in one indeterminate with their coefficients in $\mathbb{Z}_2$. In simple terms, their coefficients will be either 0 or 1. We know that this set of polynomials equipped with the usual addition and multiplication on polynomials, forms a ring, which is denoted by $\mathbb{Z}_2[x]$. Since $\mathbb{Z}_2$ is a commutative ring with unity, $\mathbb{Z}_2[x]$ is also a commutative ring with unity.

Note that the polynomials can be defined as infinite formal sums,

$$\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + \cdots + a_n x^n + \cdots,$$

where $a_i \in \mathbb{Z}_2$ and $a_i = 0$ for all except a finite number of values of i.

For example, $x^2 + x + 1$ can be written as $\sum_{i=0}^{\infty} 1.x^i$ where $a_i = 0$ for all $i > 2$.

Following this, for simplicity, we will express a polynomial of degree n as $\sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + \cdots + a_n x^n$, when $a_i = 0$ for all $i > n$.

As $\mathbb{Z}_2$ is a field, we know that for any $a, b \in \mathbb{Z}_2$ if $ab = 0$ then either $a = 0$ or $b = 0$. From this, we can conclude that $\mathbb{Z}_2[x]$ is an integral domain.

> In order to briefly describe the proof, we consider any field F and from it, any two polynomials $f(x), g(x)$ of degrees n and m such that, $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{i=0}^{m} b_i x^i$, and $a_n, b_m \neq 0$.
>
> Let us take the product of the two polynomials, $f(x)g(x)$. The coefficient of the kth term of the resulting polynomial can be written as $\sum_{i=0}^{k} a_i b_{k-i} = 0$ where $0 \leq k \leq n + m$.
>
> We see that the coefficient of the first term is $a_n b_m$. As F is a field, $a_n b_m \neq 0$. This tells us that the product of two non-zero polynomials of $F[x]$ cannot be equal to 0. Otherwise, it would lead us to a contradiction.
>
> Hence, $F[x]$ is an integral domain.

Furthermore, we can define a norm function, $\mathrm{N}: \mathbb{Z}_2 \rightarrow \mathbb{N} \cup \{0\}$ such that

$\mathrm{N}(f(x)) =$ degree of $f(x)$, where $f(x)$ is an element of $\mathbb{Z}_2$.

It is easy to establish that for any two non-zero polynomials $f(x)$ and $g(x) \in \mathbb{Z}_2$,

$$\deg(f(x)) \leq \deg(f(x)g(x)).$$

Thus, $\mathbb{Z}_2$ is a Euclidean domain and we can define a Euclidean algorithm for this ring by first establishing the division algorithm.

II. The Division algorithm and the Euclidean algorithm

For any field F, the division algorithm for the polynomials belonging to $F[x]$ can be defined as follows:

Let $f(x)$ and $g(x)$ be two elements of $F[x]$ of degrees n and m,

$f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{i=0}^{m} b_i x^i$, and $a_n, b_m \neq 0$.

Then there exists a unique pair of polynomials $q(x)$ and $r(x)$ in $F[x]$ such that,

$$f(x) = g(x).q(x) + r(x)$$

where either $r(x) = 0$ or $deg(r(x)) < deg(g(x)) = m$.

Our objective was to be able to work with relatively prime polynomials. In order to do that, it is important to learn how to find the greatest common divisor. We can define a Euclidean algorithm on $\mathbb{Z}_2$ using our knowledge of the division algorithm. (Note: every non-constant polynomial in $\mathbb{Z}_2[x]$ has '1' as its leading coefficient)

Let $f(x)$ and $g(x)$ be any two elements of $\mathbb{Z}_2[x]$, not both zero. Then there exists polynomials $s(x)$ and $t(x) \in \mathbb{Z}_2[x]$ such that,

$$s(x)f(x) + t(x)g(x) = \gcd(f(x), g(x)).$$

Here, 'gcd' stands for greatest common divisor. We define the greatest common divisor $d(x)$ to be a polynomial of the highest degree for which $d(x)$ divides $f(x)$ and $d(x)$ divides $g(x)$.

The division algorithm tells us that, suppose $f(x)$ and $g(x)$ be polynomials of degrees n and m, $n \geq m$, then there exists a unique pair of polynomials $q(x)$ and $r(x)$ such that,

$f(x) = g(x).q(x) + r(x)$ where either $r(x) = 0$ or $deg(r(x)) < deg(g(x)) = m$.

Thus, the greatest common divisor $d(x)$ of $f(x)$ and $g(x)$, divides

$$r(x) = f(x) - g(x).q(x).$$

From this we get a neat helpful relation,

$\gcd(f(x), g(x)) = \gcd(g(x), r(x))$.

We can use this simpler relation repetitively to find the greatest common divisor of any two polynomials of our chosen field. If m, the degree of polynomial $g(x)$, is at least 1 i.e., $m \geq 1$, then as the $deg(r(x)) < deg(g(x)) = m$, we know that after a finite number of repetitions, the algorithm will lead us to an ordered pair, $(c(x), k)$ where $deg(c(x)) \leq 1$ and k is a constant polynomial, i.e., $k \in \mathbb{Z}_2$.

Now, if $k = 0$, then $\gcd(f(x), g(x)) = c(x)$. Otherwise, $k = 1$ and the polynomials $f(x)$ and $g(x)$ are said to be relatively prime.

III. Polynomials in $\mathbb{Z}_2$

We will now try to identify relatively prime polynomials in $\mathbb{Z}_2$ in order to plot them and generate our quilt! For simplicity and ease, we only consider polynomials up to the third degree.

Since $\mathbb{Z}_2$ only has two elements, we can list out all the polynomials we will be working with:

$0, \quad 1,$
$x, \quad x + 1,$
$x^2, \quad x^2 + 1, \quad x^2 + x, \quad x^2 + x + 1,$
$x^3, \quad x^3 + 1, \quad x^3 + x, \quad x^3 + x + 1,$
$x^3 + x^2 + 1, \quad x^3 + x^2 + x, \quad x^3 + x^2 + x + 1.$

Suppose we were to find the greatest common divisor of $f(x) = x^3 + x^2 + x + 1$ and $g(x) = x^3 + 1$. Then, applying the Euclidean Algorithm we get this sequence of equations:

$f(x) = 1.(x^3 + 1) + (x^2 + x) = q_1(x).g(x) + r_1(x)$

$g(x) = (x + 1)(x^2 + x) + (x + 1) = q_2(x)r_1(x) + r_2(x)$

$r_1(x) = (x)(x + 1) + 0 = q_3(x)r_2(x) + 0.$

Thus, $f(x) = x^3 + x^2 + x + 1$ and $g(x) = x^3 + 1$ are not relatively prime and their greatest common divisor is $(x + 1)$.

Note that we can find the required quotient and remainder polynomials by the usual long division keeping in mind that we are in $\mathbb{Z}_2$. For example:

$$x^3 + 1 \overline{)x^3 + x^2 + x + 1}$$

$$\underline{x^3 + 1}$$

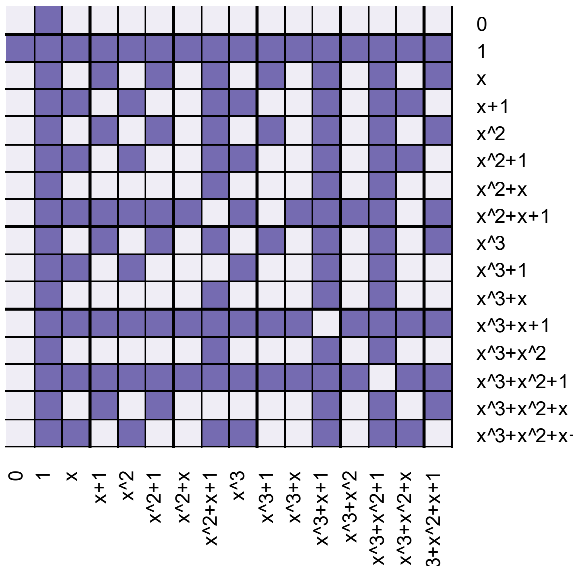
$$0 + x^2 - 1 + x + 1$$

Thus, the remainder is $x^2 - 1 + x + 1 = x^2 + x + (1 + 1) = x^2 + x$.

We can also denote this chain of equations in a simpler format:

$$(x^3 + x^2 + x + 1, x^3 + 1) \xrightarrow{q_1(x)=1} (x^3 + 1, x^2 + x) \xrightarrow{q_2(x)=(x+1)} (x^2 + x, x + 1) \xrightarrow{q_3(x)=x} (x + 1,0).$$

Hence, we take pairs of polynomials of $\mathbb{Z}_2$ and check if they are relatively prime. We use this information to form our quilt! Using the heatmap.2 function from the R package 'gplots,'we can denote every ordered pair of *relatively prime* polynomials as a purple square and every order pair of *non-relatively prime* polynomials as a white square.



This quilt does offer some insight into the relatively prime polynomials of $\mathbb{Z}_2[x]$.

If we look at the rectangle representing the ordered pairs of all first-degree polynomials, we will notice that they contain an equal number of purple and white squares. This holds for the rectangles containing the second- and third-degree polynomials as well.

Mathematicians studying this found that: if two polynomials (where both do not have degrees equal to 0), are randomly chosen from $\mathbb{Z}_2[x]$, then the probability of them being relatively prime is $\frac{1}{2}$.

IV. Probability of relatively prime polynomials in $\mathbb{Z}_2[x]$.

To understand how that can be true, we can utilise the Euclidean algorithm to match any non-relatively prime pair of polynomials $(f(x), g(x))$ with a relatively prime pair of polynomials $(f_1(x), g_1(x))$ of the same degree.

The previously mentioned theorem can be formally written as: Let $f(x)$ and $g(x)$ be two polynomials from the set of polynomials in $\mathbb{Z}_2[x]$ of degrees m and n, where both m and n are not zero. If they are chosen randomly (i.e. uniformly and independently), then the probability that $f(x)$ and $g(x)$ are relatively prime is $\frac{1}{2}$.

Suppose $n = 0$, then $g(x) = 0$ or 1. Thus, any non-relatively prime pair $(f(x), 0)$ can be matched with a relatively prime pair $(f(x),1)$.

We hope to do something similar in the general case.

Suppose $1 \leq n \leq m$, then if we applied the Euclidean algorithm on a non-relatively prime pair $\left(f(x), g(x)\right)$, we would finally arrive at $(c(x),0)$ where $c(x)$ (at least a first-degree polynomial) is the greatest common divisor of $f(x)$ and $g(x)$. It gives us a unique sequence as, by the division algorithm, we know that their exists only one pair of quotient and remainder polynomials that satisfy the conditions.

Suppose the sequence is:

$$(f(x), g(x)) \xrightarrow{q_1(x)} \left(g(x), r_1(x)\right) \xrightarrow{q_2(x)} (r_1(x), r_2(x)) \xrightarrow{q_3(x)} (c(x),0)$$

Then, due to its uniqueness we can reverse this process- starting from $(c(x),0)$ reversing the order of the quotient polynomials and ending at $\left(f(x), g(x)\right)$.

Using our previous example of the ordered pair (x^3+x^2+x+1, x^3+1), we start from its greatest common divisor:

$$(x+1,0) \xrightarrow{q_1(x)=x} (x^2+x, x+1) \xrightarrow{q_2(x)=x+1} (x^3+1, x^2+x) \xrightarrow{q_3(x)=1} (x^3+x^2+x+1, x^3+1)$$

In a similar way, we can start with the pair $(c(x), 1)$ and through the reversed order of quotient polynomials arrive at a pair of relatively prime polynomials $(f_1(x), g_1(x))$ of the same degree.

In this manner, every non-relatively prime pair $\big(f(x), g(x)\big)$ can be matched with a pair of relatively prime polynomials $(f_1(x), g_1(x))$ of the same degree.

Mathematical quilting, whether physical or digital, is a way of exploring mathematical ideas through art and creativity. As we have seen, it can combine several concepts together and even illustrate mathematical proofs in a unique way.

V. The R program:

```
#Here, 0 represents an ordered pair of polynomials that are not relatively prime while 1 represents a pair that is relatively prime.

A=matrix(c(0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0),ncol=1)
B=matrix(c(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1),ncol=1)
C=matrix(c(0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1),ncol=1)
D=matrix(c(0,1,1,0,1,0,0,1,1,0,0,1,0,1,1,0),ncol=1)
E=matrix(c(0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1),ncol=1)
F=matrix(c(0,1,1,0,1,0,0,1,1,0,0,1,0,1,1,0),ncol=1)
G=matrix(c(0,1,0,0,0,0,0,1,0,0,0,1,0,1,0,0),ncol=1)
H=matrix(c(0,1,1,1,1,1,1,0,1,0,1,1,1,1,0,1),ncol=1)
I=matrix(c(0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1),ncol=1)
J=matrix(c(0,1,1,0,1,0,0,0,1,0,0,1,0,1,0,0),ncol=1)
K=matrix(c(0,1,0,0,0,0,0,1,0,0,0,1,0,1,0,0),ncol=1)
```

```
L=matrix(c(0,1,1,1,1,1,1,1,1,1,1,0,1,1,1,1),ncol=1)

M=matrix(c(0,1,0,0,0,0,0,1,0,0,0,1,0,1,0,0),ncol=1)

N=matrix(c(0,1,1,1,1,1,1,1,1,1,1,1,1,0,1,1),ncol=1)

O=matrix(c(0,1,0,1,0,1,0,0,0,0,0,1,0,1,0,1),ncol=1)

P=matrix(c(0,1,1,0,1,0,0,1,1,0,0,1,0,1,1,0),ncol=1)

poly=cbind(A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,P)

#we rename the rows and columns accordingly

x=c("0", "1", "x",
"x+1","x^2","x^2+1","x^2+x","x^2+x+1","x^3","x^3+1","x^3+x","x^3+x
+1","x^3+x^2","x^3+x^2+1","x^3+x^2+x","x^3+x^2+x+1")

y=c("0", "1", "x",
"x+1","x^2","x^2+1","x^2+x","x^2+x+1","x^3","x^3+1","x^3+x","x^3+x
+1","x^3+x^2","x^3+x^2+1","x^3+x^2+x","x^3+x^2+x+1")

rownames(poly)=x

colnames(poly)=y

#forming the "quilt"

poly

library(gplots)

library(RColorBrewer)

heatmap.2(poly,

Rowv =NULL,

Colv=NULL,col=colorRampPalette(brewer.pal(3,"Purples"))(10),
rowsep=c(1:16), colsep=c(1:16),sepcolor="black",sepwidth=c(0.01,0.01),

key = FALSE,

density.info=c("none")

trace=c("none"))
```

A GLIMPSE INTO MERSENNE PRIMES

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Consider a number of the form $M_n = (2^n - 1)$ where $n \geq 1$. Numbers of this form are generally called Mersenne numbers, named after the 17th-century French scholar Marin Mersenne.

If the number M_n is prime, then the number is said to be a Mersenne prime.

Now, we shall see a result.

Result: let $a > 0$, $j \geq 2$.If a^j-1 is prime, then a=2 and j is also a prime.

Proof: we can write, $a^j-1 = (a-1)\ (a^{j-1} + a^{j-2} + \ldots + a + 1)$

Where, $a^{j-1} + a^{j-2} + \ldots + a + 1 \geq a+1 > 1$

Since a^j-1 is prime, the other factor must be 1. [if p is a prime, p=ab, b>1 then, b=p and a must be 1]

So, a-1=1 or, a=2

Now, let j is not prime. Then, j can be written as –

$j = kn$ where $k>1$, $n>1$

so, $a^j-1 = (a^k)^n-1$

or, $a^j-1 = (a^k-1)(\ a^{k(n-1)} + a^{k(n-2)} + \ldots + a^k + 1)$

each factor on R.H.S is greater than 1. This implies a^j-1 is not prime. This is a contradiction to the fact that a^j-1 is prime.

So, j must be prime. This completes the proof.

By the help this result, we can say that, if M_n is prime, then n is prime.

So, we can define Mersenne prime as a prime number of the form $M_p = (2^p - 1)$ where $p \geq 1$ and p is a prime number.

For an example, for p=2, $M_2 = 2^2 - 1 = 4 - 1 = 3$ which is a prime number. So, 3 is a Mersenne prime.

Now we know that M_p is prime means P is prime. But conversely can we say that if P is a prime, then M_p is prime?

The simple answer is NO!

Why? Consider M_{11}, 11 is a prime but $M_{11} = 2^{11} - 1 = 2047 = 23 \times 89$ is not a prime.

So M_{11} is composite. So, M_p is prime implies P is prime, but for any prime P, M_p is not prime.

Actually M_p is prime for p=2,3,5,7,13,17,19...and there are many more primes.

Now, we want to show a relationship between Mersenne primes and perfect numbers.

Perfect numbers: A positive integer n is said to be perfect if n is equal to the sum of all its positive divisors, excluding n itself.

For an example, 6 has positive divisors 1,2,3 and 6.

Excluding 6 we get,6=1+2+3. So 6 is a perfect number.

We take a perfect number n where σ denotes the sum of all the positive divisors of n. Then, by definition, $\sigma(n) - n = n$, or $\sigma(n) = 2n$.

For 6, $\sigma(6) = 1+2+3+6 = 12$, $\sigma(6) - 6 = 6$, or, $\sigma(6) = 2 \times 6$

Now, we shall prove a result which relates the Mersenne primes with the perfect numbers.

Result: If 2^c-1 is prime (c>1), then a= $2^{(c-1)}$ (2^c-1) is perfect and every even perfect number is of this form.

Proof:

[In this proof, we shall use the following results when needed

1. if $n=p_1^{k1}p_2^{k2}...p_r^{kr}$ is the prime factorization of n>1 then $\sigma(n)= (p_1^{k1+1}-1/p_1-1)...(p_r^{kr+1}-1/p_r-1)$

and

2. $\sigma(mn)= \sigma(m)\,\sigma(n)$ if gcd(m,n)=1]

Let, a be an even perfect number.

$a = 2^{(c-1)} n$ where n is an odd integer and $c \geq 2$

So, $\gcd(2^{(c-1)}, n)=1$

Then $\sigma(a)= \sigma(2^{(c-1)}n)= \sigma(2^{(c-1)})\,\sigma(n)= (2^c-1)\,\sigma(n)$ [It can be easily checked that $\sigma(2^{(c-1)})=2^c-1$]. Since a is perfect, $\sigma(a)=2a=2^c n$

So, $2^c n=(2^c-1)\,\sigma(n)$

This implies $2^c-1 \mid 2^c n$

But 2^c-1 and 2^c are relatively prime to each other.

So, $2^c-1|n$

Say, $n=(2^c-1)\,w$

Putting n is previous equation,

$2^c(2^c-1)\,w=(2^c-1)\,\sigma(n)$

Or $2^c w= \sigma(n)$

Now w and n, both are divisors of n, so,

$2^c w= \sigma(n)\geq w+n=w+(2^c-1)\,w=2^c w$

So, $\sigma(n)= w+n$

So, n has ony 2 positive divisors, w and n. so, n has to be a prime number and w=1.

So,n=(2^c-1) is a prime number, which completes one part of the proof.

For the other part, let (2^c-1) =p, a prime number.

We consider the integer a=(2^{c-1}) p

gcd(2^{c-1}, p)=1

$\sigma(a)= \sigma(2^{c-1}p) = \sigma(2^{c-1})\,\sigma(p)=(2^c-1)(p+1)=(2^c-1)2^c=2.(2^{(c-1)}(2^c-1))=2a$

so, a is a perfect number.

This proves our desired result.

This is known as Euclid-Euler theorem, which asserts a one-to-one correspondence between Mersenne primes and even perfect numbers.

Now we shall see some results about Mersenne primes.

According to Mersenne, M_p is prime for p=2,3,5,7,13,17,19,31,67,127,257 and composite for all other primes p<257.But, later it was proved that M_{67} is not prime.

Also first four Mersenne primes, when each one is substituted for p in the formula $2^p - 1$, a higher Mersenne prime is obtained. But that also failed for p=13.

There are various methods to check whether certain special types of Mersenne numbers are prime or composite. One such is the next one.

<u>Result</u>: If p and q=2p+1 are primes, then either $q|m_p$ or $q|M_p+2$, but not both.

<u>Proof</u>: By Fermat's theorem, 2^{q-1}=1(mod q)

Or $2^{q-1}-1 = 0(\text{mod } q)$

Or $q|2^{q-1}-1$

Now, p=(q-1)/2

$M_p=2^{(q-1)/2}-1$

$2^{q-1}-1=(2^{(q-1)/2}-1)\ (2^{(q-1)/2}+1)$

$=(2^{(q-1)/2}-1)\ (2^{(q-1)/2}-1+2)$

$=(M_p)(M_p+2)$

So, $q|(M_p)(M_p+2)$

Or $q|(M_p)$ or $q|(M_p+2)$ [as q is prime]

[we used the result p|ab implies p|a or p|b where p is a prime]

But q cannot divide both because if that happens then q will divide $(M_p+2)-(M_p)=2$, which is impossible as $q=2p+1$ or q does not divide 2.

Hence our claim is established.

Now we shall illustrate the result through a simple example.

Take p=11, q=2p+1=23 is a prime.

Then by our previous result, $23|M_{11}$ or $23|M_{11}+2$.

$M_{11}=2^{11}-1$

It is very easy to check that, $2^{11}=1 \pmod{23}$ or, $23|2^{11}-1$ or $23|M_{11}$

So, M_{11} is not prime.

Same way, we can show, M_{23} is not prime as $47|M_{23}$

Now we take M_{29}. In that case, it can be checked very easily that $59|\ M_{29}+2$.

So, now the question rises that can we get some conditions q such that $q|M_p$?

There lies our next result.

<u>Result</u>: if q= 2n+1 is a prime then

a) $q|M_n$,, if $q\equiv1 \pmod 8$ or $q\equiv7 \pmod 8$
b) $q|M_n+2$, if $q\equiv3 \pmod 8$ or $q\equiv5 \pmod 8$

As a corollary of this result, we get,

If p and q = 2p +1 are both odd primes with $p\equiv3 \pmod 4$, then $q|M_p$.

Taking p=11, we get $23|M_{11}$

Taking p=23,83, any prime satisfying this criterion... we get M_p to be composite.

Now, we shall see a couple of more results.

Result: If p is an odd prime, then any prime divisor of M_p is of the form 2kp+1

Moreover, we have another result following this:

If p is an odd prime, then any prime divisor q of M_p is of the form $q \equiv \pm 1 \pmod 8$

We now show an illustration of the above results.

We take M_{17}, 17 is a prime. So, all prime divisors will be of the form 34k+1.

$M_{17}=2^{17}-1$, $\sqrt{M_{17}}=362.037$

So, the integers of the form 34k+1 that are less than 362 are

35,69,103,137,171,205,239,273,307,341.

Among them, 103,137,239,307 are primes. It can be shown that none of them is identical to $\pm 1 \pmod 8$.

So, M_{17} has no prime divisors. So, it is a Mersenne prime.

51 Mersenne primes are known As of October 2020. The largest known prime number, $2^{82,589,933} - 1$, is a Mersenne prime. There are a lot of facts and interesting results lies in this beautiful topic. We end our discussion about the Mersenne primes here.

"Mathematics has beauty and romance. It's not a boring place to be, the mathematical world. It's an extraordinary place; it's worth spending time there".

- Marcus du Sautoy

A TECHNIQUE OF AGE DETERMINATION OF PLANATARY SURFACES

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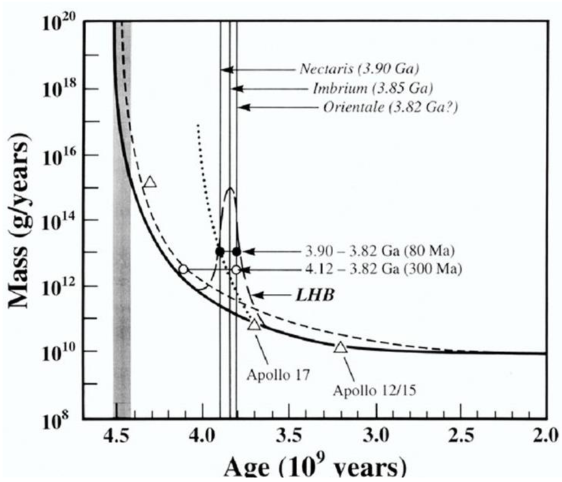
Age determination of planetary surfaces is a complex task to perform. There being no rock samples available for direct age dating, the age determination process of terrestrial planets other than our Earth gets complicated. To solve the problem, a process known as <u>Crater Chronology</u> has been developed.

From the very beginning of the formation of our solar system, meteorite impacts were frequent cataclysmic events that used to take place. The bodies, impacting on the planetary surfaces resulting in the formation of impressions or as common landforms called <u>Impact Craters</u>, are known as <u>Impactors</u>. Craters form when high velocity impactors crash on the planetary surfaces.



The frequency of impact of meteorites or asteroids on planetary surfaces was higher in the past. The crater density, i.e. the number of craters present in a unit area, is higher with increasing age of the surface. With time the number as well as size of impactors decreased. This infers that larger sized and higher crater density of a certain area is much older than that of the lower or smaller one. So, the number of craters a certain area or a certain geological feature will act as a proxy of its age.

ASSUMPTIONS:

I. Particle flux is constant over the entire planet surface
II. Frequency of the craters produced by the impactors can be measured with respect to the size of the craters.
III. Area of interest to be dated is homogenous in nature

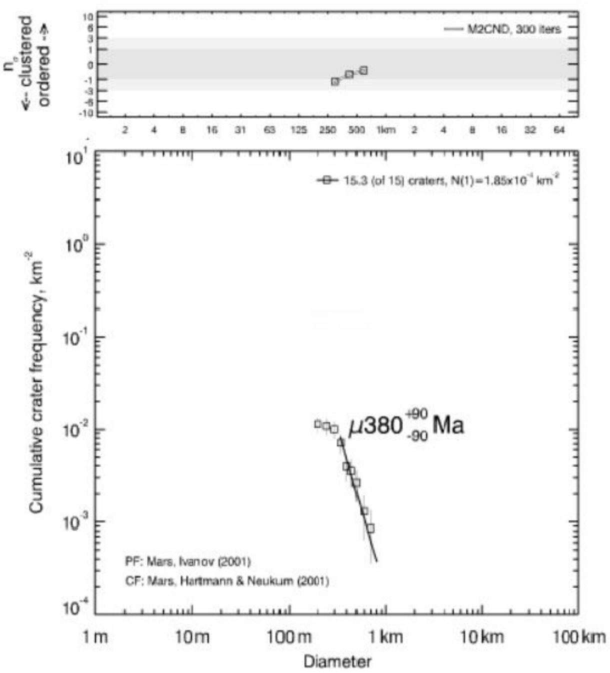
IMPORTANT TERMS:

Size-Frequency Distribution (SFD): SFD quantifies the number of craters as a function of crater size.

Production Function (PF): PF is the formation of number of craters of a particular size in relation to the number of craters of any other size.

Chronology Function (CF): It is a tool to determine the age of a certain area/surface. It is a function of SFD and PF.

SFD works only with primary craters. The factors affecting the SFD are Magmatic Flooding, Ejecta Blanketing, Secondary Cratering, Superposition, Abrasion and Infilling, Mass Wasting and Presence of Volcanic Crater.



METHOD:

The technique of Crater Chronology requires a sequence of steps to determine the relative age of any planetary surfaces. With the help of remote sensing high resolution image data and GIS applications, the entire process is executed.

The concept for Crater Chronology involves fitting the observed <u>Crater Size-Frequency Distribution</u> (CSFD) of a surface unit to a known PF, and to use the crater frequency for certain crater sizes together with a calibrating CF to obtain an absolute age. Now the age of a surface unit is to be measured. For this, it should be able to have a cumulative frequency corresponding to a standard crater diameter. That particular cumulative frequency value (Say, the standard crater diameter is taken to be 1 km) will be used to obtain the model age of the chronology function. Hence, to get that frequency the cumulative crater size-frequency distribution normalised to a unit area is plotted and then until it fits the data points the production function is shifted. Therefore, the process of age determination requires the determination of crater size frequency distribution, fitting production function and in the end estimating age from chronology function.

To execute the entire process the following steps are followed:

STEP 1: Mapping the required area and locating craters

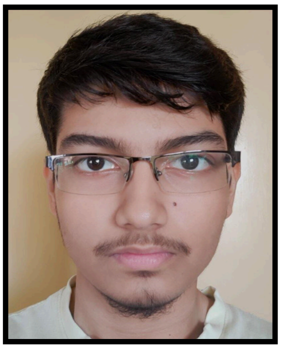
STEP 2: BUFFER CRATER COUNT

STEP 3: Randomness Analysis and Age Determination

CONCLUSION:

The craters that are to be included in the crater counting process must be primary craters, and neither any secondary nor any ghost/relic crater from an underlying older unit nor any volcanic crater can be included in the estimation. These important factors are to be kept in mind during crater counting. This CSFD technique is the only tool available to us as long return rock samples from the planets are not available to the scientists.

SATISFYING FACTORIALS



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In Mathematics, the term 'inequality' holds an important place. An inequality is a relation which makes a non-equal comparison between two numbers or other mathematical expressions. There are several kinds of inequalities. Here we are going to prove such an inequality.

We know of a famous inequality,

$$(n+1)! \geq 2^n \quad \forall\ n\ \varepsilon\ N\ \ldots\ (1)$$

We are going to use the idea of the inequality mentioned above to get an interesting outcome which is a more complicated inequality.

At first glance, the inequality that we are going to prove looks very cumbersome to the common reader, but on closer inspection, the inequality turns out to be actually true.

It is as follows:

$$(k-1)!\ k^n \leq (n+k-1)! \quad \forall\ n,\ k\ \varepsilon\ N;\ n \geq k\ \ldots\ldots(2)$$

Now we are going to prove (2).

Steps:

- We prove $(n+1)! \geq 2^n \quad \forall\ n\ \varepsilon\ N\ \ldots\ldots(1)$
- We use the idea of (1) to prove (2)

We have,

$$\begin{array}{rcl} 1 & = & 1 \\ 1\ .\ 2 & = & 1\ .\ 2 \\ 1\ .\ 2\ .\ 2 & \leq & 1\ .\ 2\ .\ 3 \\ 1\ .\ 2\ .\ 2\ .\ 2 & \leq & 1\ .\ 2\ .\ 3\ .\ 4 \\ . & & . \\ . & & . \\ . & & . \\ . & & . \end{array}$$

$$1 \;.\; 2 \;.\; 2 \ldots\ldots\ldots 2 \quad \leq \quad 1 \;.\; 2 \;.\; 3 \ldots\ldots\ldots\ldots n$$

Extending the above inequalities, we get

$$(n)! \;\geq\; 2^{n-1} \qquad \forall\; n \;\varepsilon\; N$$

Here, $n \;\varepsilon\; N$ is arbitrary.

So, we substitute n by $(n+1)$ to get,

$$(n+1)! \;\geq\; 2^{n} \qquad \forall\; n \;\varepsilon\; N$$

Similarly, we go on replacing 2 by $k \;\varepsilon\; N$ to get,

$$1 = 1$$
$$1 \;.\; 2 = 1 \;.\; 2$$
$$1 \;.\; 2 \;.\; 3 = 1 \;.\; 2 \;.\; 3$$
$$1 \;.\; 2 \;.\; 3 \;.\; 4 = 1 \;.\; 2 \;.\; 3 \;.\; 4$$

. .

. .

$$1 \;.\; 2 \;.\; 3 \;.\; 4 \ldots\ldots k = 1 \;.\; 2 \;.\; 3 \;.\; 4 \ldots\ldots k$$
$$1 \;.\; 2 \;.\; 3 \;.\; 4 \ldots\ldots\ldots k \;.\; k \leq 1 \;.\; 2 \;.\; 3 \;.\; 4 \ldots\ldots k \;.\; (k+1)$$

. .

. .

$$1 \;.\; 2 \ldots (k-1) \;.\; k \;.\; k \ldots k \leq 1 \;.\; 2 \;.\; 3 \ldots (k-1) \;.\; k \;.\; (k+1) \ldots n$$

Here, in the LHS, k is multiplied $(n-k+1)$ times,

∴ Simplifying the above inequality, we come up with its shorthand:

$$(k-1)!\; k^{n-(k-1)} \leq n! \qquad \forall\; n,\, k \;\varepsilon\; N,$$

Clearly, for $n \geq k$ the above inequality holds true.

Intuitively from the previous inequality, $n \;\varepsilon\; N$ is arbitrary.

So, we replace $n \;\varepsilon\; N$ by $n+(k-1) \;\varepsilon\; N$ to get:

$$(k-1)!\; k^{n} \leq (n+k-1)! \qquad \forall\; n,\, k \;\varepsilon\; N;\; n \geq k$$

Hence, (2) is proved.

Some useful results:

- Rearranging the above inequality we get,

$$k^{n} \,/\, n! \;\leq\; {}^{n+k-1}C_{k-1}$$

"Mathematics is not a careful march down a well-cleared highway, but a journey into a strange wilderness, where the explorers often get lost. Rigor should be a signal to the historians that the maps have been made, and the real explorers have gone elsewhere."

- W.S. Anglin

Soulful Strains

Voice for Alms

Sightless is the old beggar,
He looks up at the sky.
He makes a call to God,
He makes a call to passer-by.

A tattered gown he maintains
And a tin vessel he shakes,
Creating music for his words;
He then waits.

Loud and clear
His words, magnify lament,
Strength and unshared grief;
He stops rarely.

He calls on our mercy.
A note higher he sings,
Vibrations trembling hearts,
Ably pierce the sky.

He seldom visits busy streets,
To sing for a penny;
Every count pause his music
And he blesses one in prayer.

We overlook the street singer,
He gives his voice in exchange.
As I walk away,
The music of his performance fades.

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অপেক্ষা

সাদা কালো ফ্রেম
থেমে থাকা শহরের এলোমেলো অলিগলি,
কুয়াশা ঢাকা সকাল, স্বপ্ন দেখা আজও বাকি ;
বয়সের ছোঁয়ায় রুগ্ন চোখ দু'টি
আজও কাউকে খোঁজে ।

রাস্তার পাশের পুরোনো ল্যাম্পপোস্ট
টিমটিম করে আলো দিচ্ছে......
দাঁড়িয়ে আছে আজও
ভাঙা গড়ার স্মৃতিচিহ্ন হয়ে ।
বাগানের গোলাপগুলি, হারাচ্ছে তাদের সৌন্দর্য ;
পার্কের ধুলোয় ঢাকা বেঞ্চটি, আজও খালি ।

তবুও আকাশ পানে চেয়ে থাকা
অপেক্ষায়...........

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A Lullaby in the Dark

An existence without a face,
Walking the same road at a fluid pace;
When destiny plays its hand,
Time slips away like sand.

Who am I? What is the meaning of identities?
My world crumbles beneath my feet.
I am floating between infinite realities;
I sense none who belong to my creed.

I am everything I choose to be,
I am everybody I kill, every being I set free.
I am a product of my violence;
The mirage I decide to see through my lens.

I am water mixed with blood,
Friend or foe, who knows?
I am just playing my part;
My presence preceded by ominous crows.

I will take the form you choose
I am here for war, to make a truce
I am a concept, an intangible being
I am snow turned red, your voice when you sing.

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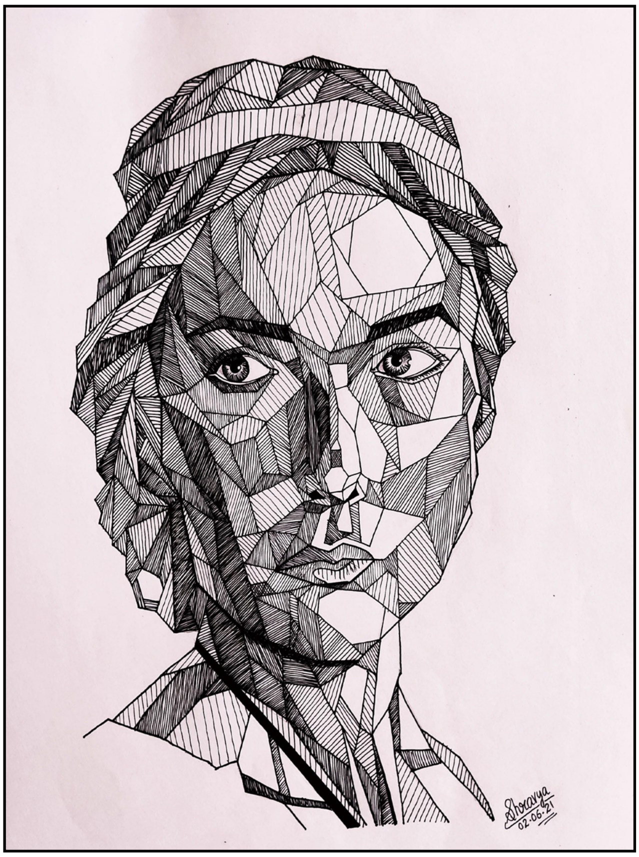
Canvas of Creation



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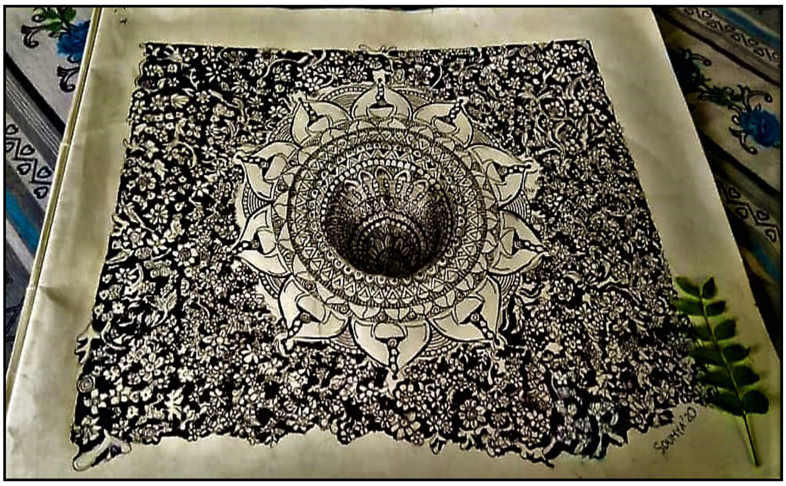

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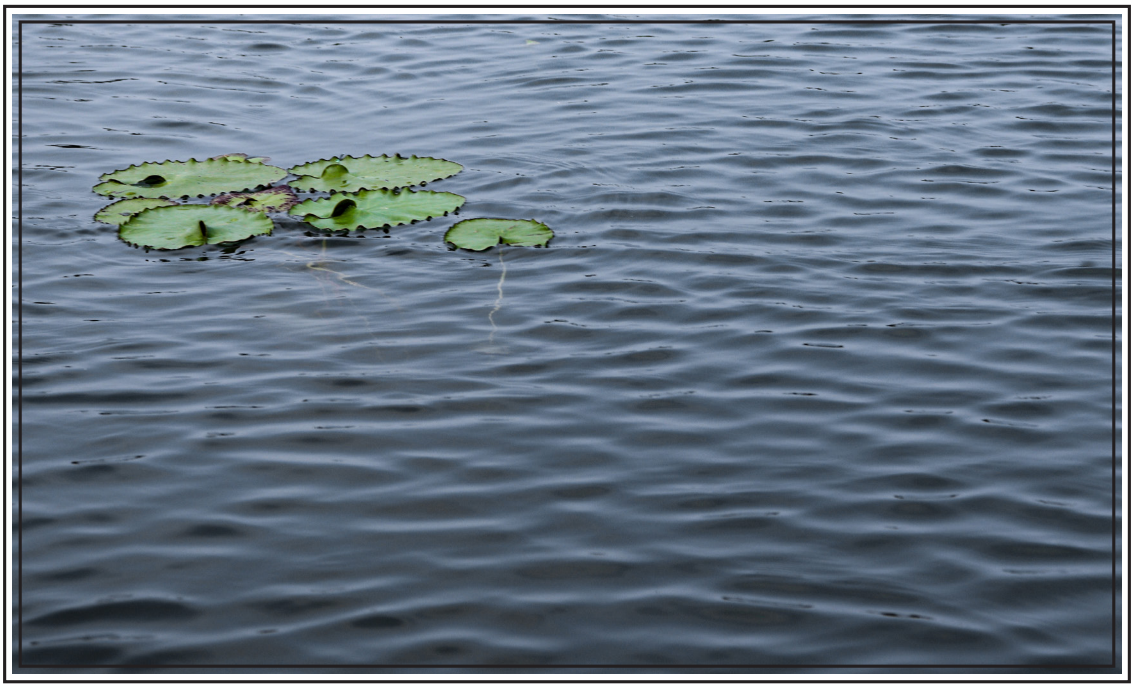

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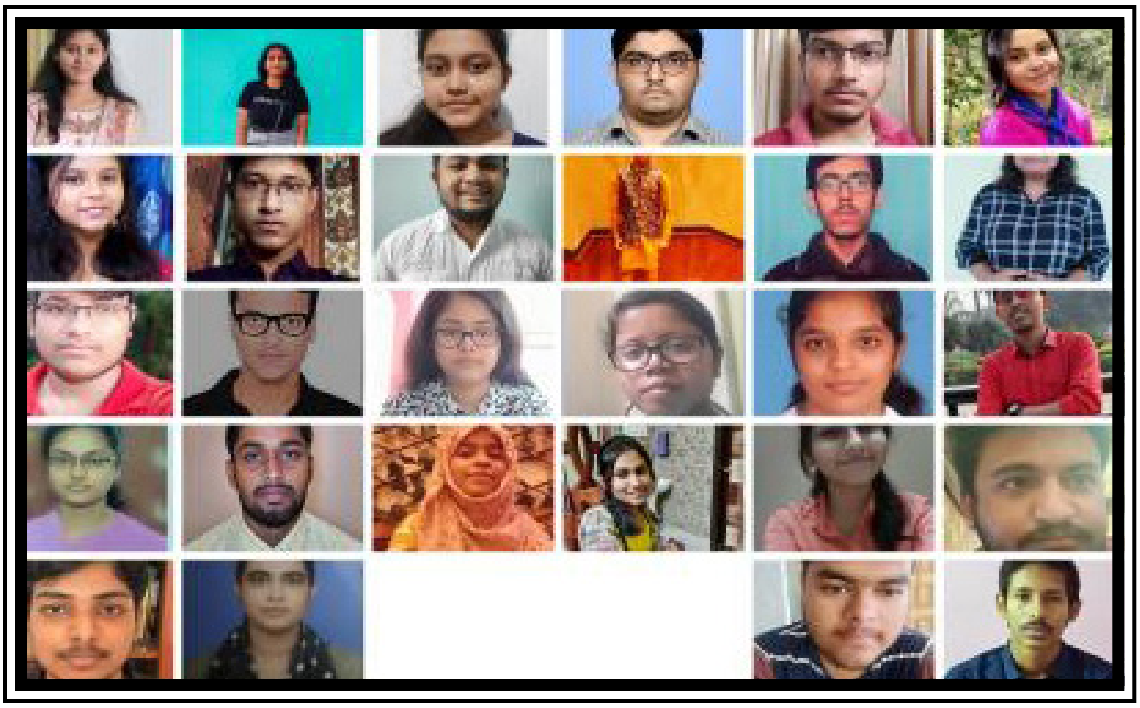
Ad Infinitum



Second Years

Third Years

Our Professors

Prof. Sucharita Roy

Dr. Tarun Kumar Bandyopadhyay

Prof. Diptiman Saha

Prof. Anindya Dey

Prof. Md. Rabiul Islam

Dr. Pabitra Debnath

Prof. Gaurab Tripathi

Crossword Puzzle

Find the Mathematicians in the below crossword puzzle corresponding to their quotes:

V	C	N	A	S	P	T	D	F	P	N	C	E	T	T
E	D	E	D	E	T	E	C	T	I	V	G	V	S	S
N	C	A	N	A	H	C	E	M	L	P	E	F	D	I
G	M	B	V	H	T	C	S	M	O	J	O	E	O	T
I	F	U	R	I	H	C	A	E	O	N	R	H	E	N
N	A	I	C	I	D	U	M	H	E	F	G	C	I	A
E	S	L	C	N	A	H	N	V	H	C	C	R	L	D
E	C	D	O	C	T	C	I	R	L	A	A	A	E	R
K	U	R	T	G	O	D	E	L	O	R	N	Z	T	A
W	E	R	T	N	R	C	H	E	B	T	T	R	E	P
E	N	V	W	E	T	L	T	R	U	E	O	I	L	J
I	T	A	E	N	I	T	P	R	C	Z	R	N	H	V
Y	Y	M	L	C	S	H	I	F	R	O	D	T	T	R
T	I	S	A	A	C	N	E	W	T	O	N	I	A	N
A	T	U	A	N	G	R	T	S	A	Y	D	C	H	A

Hints:

[1] "We must know, we will know."

[2] "We can't prove every true statement in Mathematics."

[3] "We build too many walls and not enough bridges."

[4] "This man diagonalized things to a bigger infinity."

[5] "The Imitation Game, The Enigma and War."

[6] "The Game of Life."

Brain Teasers

1. $f: R \to R$ be a function such that $(f(x)^n)$ is a polynomial for $n = 2,3,4...$
   Does it follow that f(x) is also a polynomial?

2. Let us assume that π is a positive rational (say $\frac{p}{q}, q \neq 0$). For Positive integer n, we define $f(x) = \frac{x^n(p-qx)^n}{n!}, x \in \mathbb{R}$. Again we define
   $F(x) = f(x) - f''(x) + f^{(4)}(x) - \ldots + (-1)^n f^{(2n)}(x)$
   Show that:
   a. F(0) and F(π) are integer.
   b. $\int_0^{\pi} f(x) sinx\, dx = F(0) + F(\pi)$
   c. Hence argue that π cannot be positive rational. (Think about the negative case).

3. k is a positive integer. Then find $\sum_{i=0}^{k} \binom{k+i}{i} 2^{k-i}$.

4. Let n and k be fixed positive integers, and a be an arbitrary non-negative integer. Choose a random k element subset X of {1,2,3,...,k+a} uniformly (that is all k elements subsets are chosen with same probability) and independently of X, choose random n-elements subset Y of {1,2,...,k+a+n} uniformly. Prove that, the probability $P\ (min(Y) > max(X))$ is independent of a.

5. Let S denote the set of all primitives of a function $f: R \to R$. Show that M along with the operation $*: S^2 \to S$ defined as $F * G = F + G(2021)$ forms an abelian group and it is isomorphic to the additive group of real numbers.

6. Suppose that f is a function on the interval [1, 3] such that $-1 \leq f(x) \leq 1$ for all and $\int_1^3 f(x)dx = 0$. Find the supremum of $\int_1^3 \frac{f(x)}{x} dx$.

7. Show that every sequence $a_1, a_2, \ldots, a_{mn+1}$ of $mn+1$ distinct real numbers contains either an increasing subsequence of length $m+1$ or a decreasing subsequence of length $n+1$.

8. Find the number of ways 66 identical coins can be separated into three nonempty piles so that there are fewer coins in the first pile than in the second pile and fewer coins in the second pile than in the third pile.

9. The repeating decimals $0.abab\overline{ab}$ and $0.abcabc\overline{abc}$ satisfy $0.abab\overline{ab} + 0.abcabc\overline{abc} = \frac{33}{37}$ where a, b, and c are (not necessarily distinct) digits. Find the three digit number abc.

10. Let $A \in M_n(C)$ be an upper triangular matrix. And $D \in M_n(R)$ is a diagonal matrix with non negative entries. Show that if $T^*T = D\ and\ TD = DT$, and then T is also diagonal.

Answer of Crossword Puzzle

V	C	N	A	S	P	T	D	F	P	N	C	E	T	T
E	[1]D	E	D	E	T	E	C	T	I	V	[4]G	V	S	S
N	C	A	N	A	H	C	E	M	L	P	E	F	D	I
G	M	B	V	H	T	C	S	M	O	[6]J	O	E	O	T
I	F	U	R	I	H	C	A	E	O	N	R	H	E	N
N	A	I	C	I	D	U	M	H	E	F	G	C	I	[5]A
E	S	L	C	N	A	H	N	V	H	C	C	R	L	D
E	C	D	O	C	T	C	I	R	L	A	A	A	E	R
[2]K	U	R	T	G	O	D	E	L	O	R	N	Z	T	A
W	E	R	T	N	R	C	H	E	B	T	T	R	E	P
E	N	V	W	E	T	L	T	R	U	E	O	I	L	J
I	T	A	E	N	I	T	P	R	C	Z	R	N	H	V
Y	Y	M	L	C	S	H	I	F	R	O	D	T	T	R
T	[3]I	S	A	A	C	N	E	W	T	O	N	I	A	N
A	T	U	A	N	G	R	T	S	A	Y	D	C	H	A

Answers:

[1] **David Hilbert**
[2] **Kurt Godel**
[3] **Isaac Newton**
[4] **Georg Cantor**
[5] **Alan Turing**
[6] **John Conway**

βεαcθn

To be continued . . .